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Numerical Solution of Unsteady Couette Flow for a Discrete Velocity Gas

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ABSTRACT
In this work we use a numerical scheme based on fractional step method to solve the initial-boundary value problem arising from the modeling of the plane Couette flow by the eight velocity spatial Broadwell model. We show that the scheme is convergent and we perform a comparison with an exact solution. A good agreement is observed.
KEYWORDS: *Boltzmann Equation, Couette Flow, Rarefied Gas, Fractional Step Method*

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I. INTRODUCTION

The plane Couette flow is the flow of a gas between two parallel moving plates. In transitional or slip flow regimes the study of Couette flows deserves the resolution of the Boltzmann equation. However the complexity of the Boltzmann equation leads to develop simpler models having its main properties. Among those the discrete kinetic models have interesting conceptual and mathematical features. The aim of this work is to present a numerical scheme based on the fractional step method to compute the solution of the problem for the eight velocity model of Broadwell [1]. The paper is organised as follow : in section II we state the physical problem, the scheme is then described and its numerical convergence is proved in the section III, then we present some numerical results in section IV and end with a comparison with the exact solution in section V.

II. STATEMENT OF THE PROBLEM

The physical space is related to the orthonormal reference R = (O, **x′** , **y′** , **z′**). The plates are located at y' = −h/2 and y′ = h/2, (h > 0). The velocities of the eight spatial velocity Broadwell model in the basis basis (**x′** , **y′** , **z′**) are $\mathbf{u}_1$ = c(−1, 1, 1), $\mathbf{u}_2$ = c(1, 1, 1), $\mathbf{u}_3$ = c(−1, −1, 1), $\mathbf{u}_4$ = c(1, −1, 1), $\mathbf{u}_5$ = c(−1, 1, −1), $\mathbf{u}_6$ = c(1, 1, −1), $\mathbf{u}_7$ = c(−1, −1, −1), $\mathbf{u}_8$ = c(1, −1, −1), where c is an arbitrary constant. We assume that the flow depends only on the space variable y ′ and the time t ′ . We denote by N_i (t ′ , y ′) the number density of particles of velocity $\mathbf{u}_i$ in point M(y ′) at time t ′. The kinetic equations of this model with binary collisions are the equations (1.1)-(1.8) [2]:

$$\frac{\partial N_1}{\partial t'} + c\frac{\partial N_1}{\partial y'} = cs\sqrt{2}(N_2N_3 - N_1N_4 + N_2N_5 - N_1N_6 + N_3N_5 - N_1N_7) + \frac{cs\sqrt{3}}{2}(N_2N_7 + N_3N_6 + N_4N_5 - 3N_1N_8), \tag{1.1}$$

$$\frac{\partial N_2}{\partial t'} + c\frac{\partial N_2}{\partial y'} = cs\sqrt{2}(N_1N_4 - N_2N_3 + N_1N_6 - N_2N_5 + N_4N_6 - N_2N_8) + \frac{cs\sqrt{3}}{2}(N_1N_8 + N_3N_6 + N_4N_5 - 3N_2N_7), \tag{1.2}$$

$$\frac{\partial N_3}{\partial t'} - c\frac{\partial N_3}{\partial y'} = cs\sqrt{2}(N_1N_4 - N_2N_3 + N_1N_7 - N_3N_5 + N_4N_7 - N_3N_8) + \frac{cs\sqrt{3}}{2}(N_2N_7 + N_1N_8 + N_4N_5 - 3N_3N_6), \tag{1.3}$$

$$\frac{\partial N_4}{\partial t'} - c\frac{\partial N_4}{\partial y'} = cs\sqrt{2}(N_2N_3 - N_1N_4 + N_2N_8 - N_4N_6 + N_3N_5 - N_4N_7) + \frac{cs\sqrt{3}}{2}(N_2N_7 + N_3N_6 + N_1N_8 - 3N_4N_5), \quad (1.4)$$

$$\frac{\partial N_5}{\partial t'} + c\frac{\partial N_5}{\partial y'} = cs\sqrt{2}(N_1N_6 - N_2N_5 + N_1N_7 - N_3N_5 + N_6N_7 - N_5N_8) + \frac{cs\sqrt{3}}{2}(N_2N_7 + N_3N_6 + N_1N_8 - 3N_4N_5), \quad (1.5)$$

$$\frac{\partial N_6}{\partial t'} + c\frac{\partial N_6}{\partial y'} = cs\sqrt{2}(N_2N_5 - N_1N_6 + N_2N_8 - N_4N_6 + N_5N_8 - N_6N_7) + \frac{cs\sqrt{3}}{2}(N_2N_7 + N_1N_8 + N_4N_5 - 3N_3N_6), \quad (1.6)$$

$$\frac{\partial N_7}{\partial t'} - c\frac{\partial N_7}{\partial y'} = cs\sqrt{2}(N_5N_8 - N_6N_7 + N_3N_5 - N_1N_7 + N_3N_8 - N_4N_7) + \frac{cs\sqrt{3}}{2}(N_1N_8 + N_3N_6 + N_4N_5 - 3N_2N_7), \quad (1.7)$$

$$\frac{\partial N_8}{\partial t'} - c\frac{\partial N_8}{\partial y'} = cs\sqrt{2}(N_4N_6 - N_2N_8 + N_4N_7 - N_3N_8 + N_6N_7 - N_5N_8) + \frac{cs\sqrt{3}}{2}(N_2N_7 + N_3N_6 + N_4N_5 - 3N_1N_8). \quad (1.8)$$

We assume that $N_1 = N_5$, $N_2 = N_6$, $N_3 = N_7$, $N_4 = N_8$ according to the symmetry of the model and that of the physical problem. The macroscopic variables of the flow are the mean density N , the longitudinal velocity U and the transversal velocity V given by :

$$N = 2(N_1 + N_2 + N_3 + N_4),$$
$$NU = 2c(-N_1 + N_2 - N_3 + N_4), \quad (2)$$
$$NV = 2c(N_1 + N_2 - N_3 - N_4).$$

The Maxwellian densities of the model associated with the macroscopic variables N, U and V are:

$$N_{1M} = \frac{N}{8}\left(1-\frac{U}{c}\right)\left(1+\frac{V}{c}\right), \quad N_{2M} = \frac{N}{8}\left(1+\frac{U}{c}\right)\left(1+\frac{V}{c}\right),$$
$$N_{3M} = \frac{N}{8}\left(1-\frac{U}{c}\right)\left(1-\frac{V}{c}\right), \quad N_{4M} = \frac{N}{8}\left(1+\frac{U}{c}\right)\left(1-\frac{V}{c}\right). \quad (3)$$

The microscopic densities of the discrete gas in Maxwellian equilibrium with a wall, denoted $N_{lw}^{\pm}$ are the Maxwellian densities associated with 1 and the longitudinal and transversal velocities of the wall respectively denoted by $U_w^{\pm}$ and $V_w^{\pm}$. Let $\lambda^{\pm}$ the respective accommodation coefficients. The boundary conditions of diffuse reflection [2, 3] are :

$$N_1(t',-h/2)=\lambda^-(t')N_{1w}^- = \frac{\lambda^-(t')}{8}\left(1-\frac{U_w{}^-}{c}\right)\left(1+\frac{V_w{}^-}{c}\right),$$
$$N_2(t',-h/2)=\lambda^-(t')N_{2w}^- = \frac{\lambda^-(t)}{8}\left(1+\frac{U_w{}^-}{c}\right)\left(1+\frac{V_w{}^-}{c}\right),$$
$$N_3(t',h/2)=\lambda^+(t')N_{3w}^+ = \frac{\lambda^+(t')}{8}\left(1-\frac{U_w{}^+}{c}\right)\left(1-\frac{V_w{}^+}{c}\right),$$
$$N_4(t',h/2)=\lambda^+(t')N_{4w}^+ = \frac{\lambda^+(t')}{8}\left(1+\frac{U_w{}^+}{c}\right)\left(1-\frac{V_w{}^+}{c}\right). \quad (4)$$

The impermeability of the plates means that the normal velocity near the plates vanishes. Therefore :

$$\mathbf{U}^- \cdot \mathbf{n}^- = 0, \qquad \mathbf{U}^+ \cdot \mathbf{n}^+ = 0 \quad (5)$$

where $\mathbf{n}^-$ and $\mathbf{n}^+$ denote the inward-pointing (i.e. into the gas) unit vectors normal to the plates and $\mathbf{U}^-$ and $\mathbf{U}^+$ the velocities of the discrete gas at M(-h/2) and M(h/2) respectively. We assume that initially the gas is in the Maxwellian state with a total density N_0 , a longitudinal and transverse velocity respectively U_0 and V_0. We have:

$$N_1(0,y')=N_1^0=\frac{N_0}{8}\left(1-\frac{U_0}{c}\right)\left(1+\frac{V_0}{c}\right),\ \ N_2(0,y')=N_2^0=\frac{N_0}{8}\left(1+\frac{U_0}{c}\right)\left(1+\frac{V_0}{c}\right)$$
$$N_3(0,y')=N_3^0=\frac{N_0}{8}\left(1-\frac{U_0}{c}\right)\left(1-\frac{V_0}{c}\right),\ \ N_4(0,y')=N_4^0=\frac{N_0}{8}\left(1+\frac{U_0}{c}\right)\left(1-\frac{V_0}{c}\right). \quad (6)$$

We choose the reference quantities N_0 , h and c respectively for the density, the lenght and the velocity and introduce the following dimensionless variables and parameters :

$$y=y'/h, t=ct'/h,\ Kn=(sN_0h)^{-1}, n_l=N_l/N_0, n_{lw}^{\pm}=N_{lw}^{\pm},\ n_l^0=N_l^0/N_0$$
$$u_w^{\pm}=U_w^{\pm}/c, v_w^{\pm}=V_w^{\pm}/c, u_0=U_0/c, v_0=V_0/c,\ u=U/c,\ v=V/c. \quad (7)$$

The problem is put in the dimensionless form :

$$\frac{\partial n_1}{\partial t}+\frac{\partial n_1}{\partial y}=\frac{\sqrt{2}+\sqrt{3}}{Kn}(n_2n_3-n_1n_4),$$
$$\frac{\partial n_2}{\partial t}+\frac{\partial n_2}{\partial y}=\frac{\sqrt{2}+\sqrt{3}}{Kn}(n_1n_4-n_2n_3),$$
$$\frac{\partial n_3}{\partial t}-\frac{\partial n_3}{\partial y}=\frac{\sqrt{2}+\sqrt{3}}{Kn}(n_1n_4-n_2n_3),$$
$$\frac{\partial n_4}{\partial t}-\frac{\partial n_4}{\partial y}=\frac{\sqrt{2}+\sqrt{3}}{Kn}(n_2n_3-n_1n_4)$$
$$n_l(0,y)=n_l^0,\quad y\in[-1/2,1/2],\ l=1,2,3,4 \quad (8)$$
$$n_1(t,-1/2)=\lambda^-(t)n_{1w}^-,\quad n_2(t,-1/2)=\lambda^-(t)n_{2w}^-,$$
$$n_3(t,1/2)=\lambda^+(t)n_{3w}^+,\quad n_4(t,1/2)=\lambda^+(t)n_{4w}^+,$$
$$n_1(t,-1/2)+n_2(t,-1/2)-n_3(t,-1/2)-n_4(t,-1/2)=0,$$
$$n_1(t,1/2)+n_2(t,1/2)-n_3(t,1/2)-n_4(t,1/2)=0.$$

III. NUMERICAL SCHEME

The numerical scheme used to solve the problem is based on fractional step method. First the problem is solved in spatial homogeneous flow (equations (9)), and secondly it is solved in free molecular regime (equations (10)). The time step is Δt and n_l^m is the density n_l at time $t=m\Delta t$, (m=0,1,2,...), $n_l^{m+1/2}$ the density in the middle time :

$$\begin{aligned}
\frac{n_1^{m+1/2}-n_1^m}{\Delta t} &= \frac{\sqrt{2}+\sqrt{3}}{Kn}\left(n_2^{m+1/2}n_3^{m+1/2}-n_1^{m+1/2}n_4^{m+1/2}\right), \quad (9.1)\\
\frac{n_2^{m+1/2}-n_2^m}{\Delta t} &= \frac{\sqrt{2}+\sqrt{3}}{Kn}\left(n_1^{m+1/2}n_4^{m+1/2}-n_2^{m+1/2}n_3^{m+1/2}\right), \quad (9.2)\\
\frac{n_3^{m+1/2}-n_3^m}{\Delta t} &= \frac{\sqrt{2}+\sqrt{3}}{Kn}\left(n_1^{m+1/2}n_4^{m+1/2}-n_2^{m+1/2}n_3^{m+1/2}\right), \quad (9.3)\\
\frac{n_4^{m+1/2}-n_4^m}{\Delta t} &= \frac{\sqrt{2}+\sqrt{3}}{Kn}\left(n_2^{m+1/2}n_3^{m+1/2}-n_1^{m+1/2}n_4^{m+1/2}\right), \quad (9.4)
\end{aligned} \tag{9}$$

$$\begin{aligned}
\frac{n_1^{m+1}-n_1^{m+1/2}}{\Delta t}+\frac{\partial n_1^{m+1}}{\partial y} &= 0, \quad (10.1)\\
\frac{n_2^{m+1}-n_2^{m+1/2}}{\Delta t}+\frac{\partial n_2^{m+1}}{\partial y} &= 0, \quad (10.2),\\
\frac{n_3^{m+1}-n_3^{m+1/2}}{\Delta t}-\frac{\partial n_3^{m+1}}{\partial y} &= 0, \quad (10.3)\\
\frac{n_4^{m+1}-n_4^{m+1/2}}{\Delta t}-\frac{\partial n_4^{m+1}}{\partial y} &= 0, \quad (10.4)
\end{aligned} \tag{10}$$

$$\begin{aligned}
&n_{2w}^- n_1^{m+1} - n_{1w}^- n_2^{m+1} = 0, \ y=-1/2,\\
&n_{4w}^+ n_3^{m+1} - n_{3w}^+ n_4^{m+1} = 0, \ y=1/2,\\
&v^{m+1}=0, \quad y=\pm 1/2.
\end{aligned} \tag{11}$$

The equations (11) are the boundary conditions taken at the time $t = (m + 1)\Delta t$. After an explicit computation we deduce from equations (9) :

$$\begin{aligned}
n_1^{m+1/2} &= \frac{n_1^m+\alpha\left(n_1^m+n_2^m\right)\left(n_1^m+n_3^m\right)}{1+\alpha\left(n_1^m+n_2^m+n_3^m+n_4^m\right)}, \quad n_2^{m+1/2} = \frac{n_2^m+\alpha\left(n_1^m+n_2^m\right)\left(n_2^m+n_4^m\right)}{1+\alpha\left(n_1^m+n_2^m+n_3^m+n_4^m\right)},\\
n_3^{m+1/2} &= \frac{n_3^m+\alpha\left(n_1^m+n_3^m\right)\left(n_3^m+n_4^m\right)}{1+\alpha\left(n_1^m+n_2^m+n_3^m+n_4^m\right)}, \quad n_4^{m+1/2} = \frac{n_4^m+\alpha\left(n_2^m+n_4^m\right)\left(n_3^m+n_4^m\right)}{1+\alpha\left(n_1^m+n_2^m+n_3^m+n_4^m\right)},
\end{aligned} \tag{12}$$

where $\alpha = \left(\sqrt{2}+\sqrt{3}\right)\Delta t/Kn$. The quantities n_l^m and $n_l^{m+1/2}$ depend upon y. We perform a regular grid of the domain [-1/2,1/2] with the step $\Delta y = 1/(K - 1)$ where $K \in IN \setminus \{0, 1\}$. Let $n_{l,k}^{m+1}$ be the value of n_l^{m+1} at the point $y_k \in [-1/2, 1/2]$. One has:

$$\begin{aligned}
&\frac{n_{1,k}^{m+1}-n_{1,k}^{m+1/2}}{\Delta t}+\frac{n_{1,k}^{m+1}-n_{1,k-1}^{m+1}}{\Delta y}=0, \quad k=2,\cdots,K\\
&\frac{n_{2,k}^{m+1}-n_{2,k}^{m+1/2}}{\Delta t}+\frac{n_{2,k}^{m+1}-n_{2,k-1}^{m+1}}{\Delta y}=0, \quad k=2,\cdots,K\\
&\frac{n_{3,k}^{m+1}-n_{3,k}^{m+1/2}}{\Delta t}-\frac{n_{3,k+1}^{m+1}-n_{3,k}^{m+1}}{\Delta y}=0, \quad k=1,\cdots,K-1\\
&\frac{n_{4,k}^{m+1}-n_{4,k}^{m+1/2}}{\Delta t}-\frac{n_{4,k+1}^{m+1}-n_{4,k}^{m+1}}{\Delta y}=0, \quad k=1,\cdots,K-1
\end{aligned} \tag{13}$$

$$\begin{aligned}
&n_{2w}^{-}n_{1,1}^{m+1}-n_{1w}^{-}n_{2,1}^{m+1}=0,\\
&n_{4w}^{+}n_{3,K}^{m+1}-n_{3w}^{+}n_{4,K}^{m+1}=0,\\
&n_{1,1}^{m+1}+n_{2,1}^{m+1}-n_{3,1}^{m+1}-n_{4,1}^{m+1}=0,\\
&n_{1,K}^{m+1}+n_{2,K}^{m+1}-n_{3,K}^{m+1}-n_{4,K}^{m+1}=0.
\end{aligned} \tag{14}$$

Consistency

We obtain by addition of the equation (9.1) and the equations (10.1):

$$\frac{n_{1,k}^{m+1}-n_{1,k}^{m}}{\Delta t}+\frac{n_{1,k}^{m+1}-n_{1,k-1}^{m+1}}{\Delta y}=\frac{\sqrt{2}+\sqrt{3}}{Kn}\left(n_2^{m+1/2}n_3^{m+1/2}-n_1^{m+1/2}n_4^{m+1/2}\right). \tag{15}$$

Making a Taylor serie expansion we have :

$$\begin{aligned}
&\frac{n_{1,k}^{m+1}-n_{1,k}^{m}}{\Delta t}=\frac{\partial n_1}{\partial t}\left(t_{m+1},y_k\right)+O(\Delta t),\\
&\frac{n_{1,k}^{m+1}-n_{1,k-1}^{m+1}}{\Delta y}=\frac{\partial n_1}{\partial y}\left(t_{m+1},y_k\right)+O(\Delta y).
\end{aligned} \tag{16}$$

Then

$$\left(\frac{n_{1,k}^{m+1}-n_{1,k}^{m}}{\Delta t}+\frac{n_{1,k}^{m+1}-n_{1,k-1}^{m+1}}{\Delta y}\right)-\left(\frac{\partial n_1}{\partial t}\left(t_{m+1},y_k\right)+\frac{\partial n_1}{\partial y}\left(t_{m+1},y_k\right)\right)=O(\Delta t+\Delta y). \tag{17}$$

The same argument holds for $l\in\{2,3,4\}$. We thus conclude that the scheme is accurate of order 1 in time and space.

Stability

We use Fourier analysis to study the stability of the scheme. We put :

$$\begin{aligned}
&n_{l,k}^{m}=\tilde{n}_l^{m}(\eta)\exp(i\eta k\Delta y),\\
&\rho_k^m=2\sum_{l=1}^{4}n_{l,k}^{m}=\tilde{\rho}^{m}(\eta)\exp(i\eta k\Delta y)
\end{aligned} \tag{18}$$

with $\tilde{\rho}^{m}(\eta)=2\left(\tilde{n}_1^{m}(\eta)+\tilde{n}_2^{m}(\eta)+\tilde{n}_3^{m}(\eta)+\tilde{n}_4^{m}(\eta)\right)$, where η is an arbitrary wave number and i is the complex number such that $i^2=-1$. The boundedness of $\tilde{n}_l^{m}(\eta)$, $l\in\{1,2,3,4\}$ is equivalent to that of $\tilde{\rho}^{m}(\eta)$. Using the conservation of mass in equations (10), one can write $\tilde{\rho}^{m+1/2}(\eta)=\tilde{\rho}^{m}(\eta)$. We have :

$$\begin{aligned}
&n_{l,k-1}^{m}=n_{l,k}^{m}\exp(-i\eta\Delta y),\\
&n_{l,k+1}^{m}=n_{l,k}^{m}\exp(i\eta\Delta y).
\end{aligned} \tag{19}$$

We replace these relations in the equations (13) to get:

$$F_l(\eta) n_{l,k}^m = n_{l,k}^{m+1/2}, \quad l \in \{1,2,3,4\} \tag{20}$$

with

$$\begin{aligned} F_1(\eta) = F_2(\eta) = 1 + \sigma - \sigma \exp(-i\eta\Delta y), \\ F_3(\eta) = F_4(\eta) = 1 + \sigma - \sigma \exp(i\eta\Delta y). \end{aligned} \tag{21}$$

and $\sigma = \Delta t / \Delta y$. By taking the modulus, we can write :

$$\begin{aligned} |F_1(\eta)|^2 = |F_2(\eta)|^2 = [1 + \sigma(1 - \cos(\eta\Delta y))]^2 + [\sigma \sin(\eta\Delta y)]^2, \\ |F_3(\eta)|^2 = |F_4(\eta)|^2 = [1 + \sigma(1 - \cos(\eta\Delta y))]^2 + [-\sigma \sin(\eta\Delta y)]^2. \end{aligned} \tag{22}$$

For any $X \in IR, 1 - \cos(X) \geq 0$, then $|F_l(\eta)| > 1$, $l \in \{1,2,3,4\}$. Thus all the amplification factors $1/F_l(\eta)$ satisfy $|1/F_l(\eta)| < 1$. Then:

$$n_l^{m+1}(\eta) \leq n_l^{m+1/2}(\eta), \quad l \in \{1,2,3,4\}. \tag{23}$$

We obtain by addition :

$$\begin{aligned} \rho^{m+1}(\eta) &\leq \rho^{m+1/2}(\eta), \quad \forall m \\ &\leq \rho^m(\eta), \quad \forall m \end{aligned} \tag{24}$$

Finally

$$\rho^m(\eta) \leq \rho^0(\eta), \quad \forall m. \tag{25}$$

We can then conclude to the stability of the scheme and therefore to its convergence.

IV. NUMERICAL RESULTS

For the computations we put $u_w^- = -u_w^+ = -0.2$ and $v_w^\pm = 0$. The time step is Δt = 0.001 and K = 21. The transverse velocity vanishes in the flow. The longitudinal velocity profile is linear (Figure 1a) . When Kn tends towards zero the non slip condition is obtained. The velocity slip tends towards zero for Kn tending towards zero and tends to a constant value when Kn tends towards +∞ (Figure 1b).

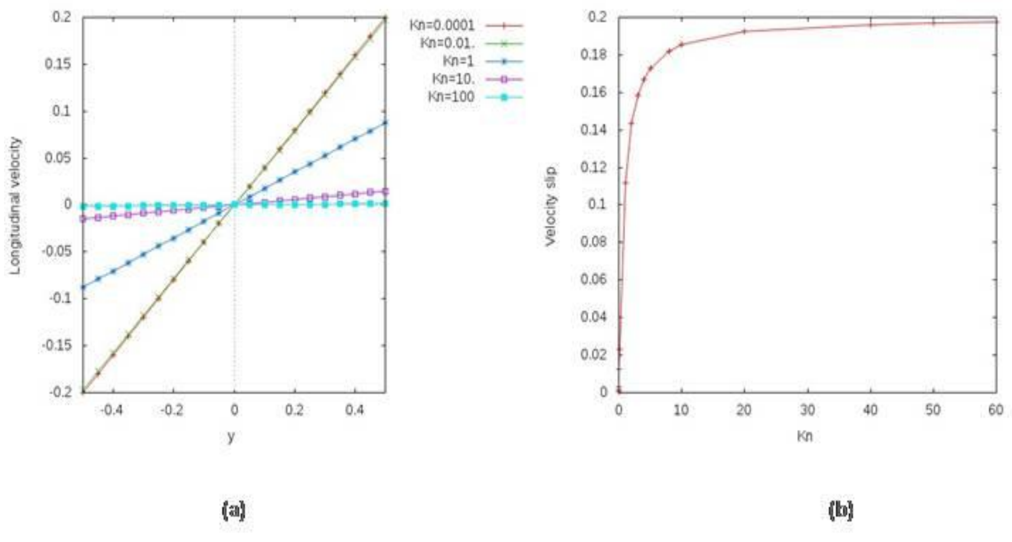


Figure 1 Velocity profile at the steady state: (a) Longitudinal velocity at the steady state (b) Velocity slip

V. COMPARISON WITH EXACT ANALYTICAL SOLUTION

To validate the scheme, we compare the numerical result to the exact analytical solution. We then consider a Couette flow between two parallel plates and we solve the boundary value problem in the steady state. The notation are the same as in the above. The problem stated is :

$$
\begin{aligned}
&\frac{\partial n_1}{\partial y} = \frac{\sqrt{2}+\sqrt{3}}{Kn}\left(n_2 n_3 - n_1 n_4\right),\\
&\frac{\partial n_2}{\partial y} = \frac{\sqrt{2}+\sqrt{3}}{Kn}\left(n_1 n_4 - n_2 n_3\right),\\
&-\frac{\partial n_3}{\partial y} = \frac{\sqrt{2}+\sqrt{3}}{Kn}\left(n_1 n_4 - n_2 n_3\right),\\
&-\frac{\partial n_4}{\partial y} = \frac{\sqrt{2}+\sqrt{3}}{Kn}\left(n_2 n_3 - n_1 n_4\right)\\
&n_1(-1/2) = \lambda^-\left(1-u_w^-\right)/8, \quad n_2(-1/2) = \lambda^-\left(1+u_w^-\right)/8,\\
&n_3(1/2) = \lambda^+\left(1-u_w^+\right)/8, \quad n_4(1/2) = \lambda^+\left(1+u_w^+\right)/8,\\
&n_1(-1/2)+n_2(-1/2)-n_3(-1/2)-n_4(-1/2)=0,\\
&n_1(1/2)+n_2(1/2)-n_3(1/2)-n_4(1/2)=0.
\end{aligned}
\tag{26}
$$

The exact analytical solution $\left(n_1, n_2, n_3, n_4\right)$ of this equations is given by :

$$n_1(y) = \frac{\beta k_2}{16}y + k_1, \quad n_2(y) = \frac{1}{4} - n_1(y), \quad n_3(y) = \frac{k_2}{4} + n_1(y), \quad n_4(y) = \frac{1}{4} - \frac{k_2}{4} + n_1(y), \tag{27}$$

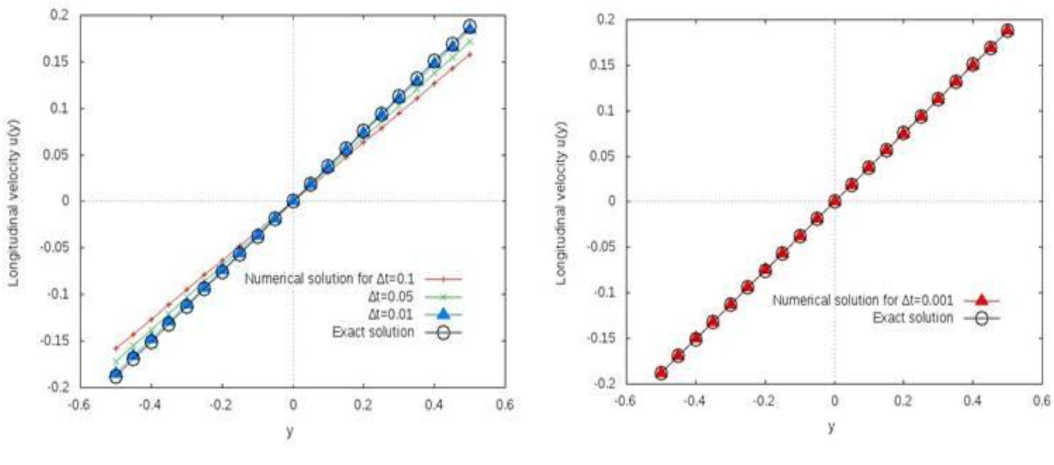
where

$$k_1 = \frac{1-u_w^-}{8} + \frac{\beta\left(u_w^- - u_w^+\right)}{16\beta+64}, \quad k_2 = \frac{2\left(u_w^- - u_w^+\right)}{\beta+4}, \quad \beta = \frac{\sqrt{2}+\sqrt{3}}{Kn}. \tag{28}$$

Then the longitudinal velocity u is :

$$u(y) = 2\left[-n_1(y)+n_2(y)-n_3(y)+n_4(y)\right] = \frac{\beta\left(u_w^+ - u_w^-\right)}{\beta+4}y + \frac{u_w^- + u_w^+}{2}. \tag{29}$$

We find a good agreement of the exact and numerical results as shown on the figure 2 and in the table 1.



(a) (b)

Figure 2 Comparison with exact analytical solution

Y	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
Exact solution	0	0.0188	0.0376	0.0564	0.0752	0.0940	0.1128	0.1317	0.1504	0.1692	0.1880
Numerical solution (Δt = 0.001)	0	0.0188	0.0375	0.0563	0.0751	0.0938	0.1126	0.1314	0.1502	0.1689	0.1877

Table 1 Comparison value

VI. CONCLUSION

We solve the unsteady Couette flow problem by means of a scheme based on fractional step method. The scheme converge and we find a good agreement with exact solution. We show the influence of the time step on the accuracy of the scheme.

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