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Generalized Soft Group Homomorphisms

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ABSTRACT
The aim of this paper is to introduce the notions of generalized soft homomorphism of generalized soft groups, generalized soft (inverse) image of generalized soft (normal) subgroup under generalized soft homomorphism, generalized soft kernel of a generalized soft homomorphism etc., generalizing the corresponding existing notions for a soft group over a group and show that several of the crisp theoretic results naturally extend to these new objects too.

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I. INTRODUCTION

Ever since Molodtsov[3] introduced the concept of a soft set over a universal set as an alternative in the theories of uncertainties, several mathematicians started imposing and studying algebraic, topological and topologically algebraic structures on them. In fact, a soft set is nothing but pair (F, E) where $F: E \to P(U)$ is a mapping, E is a set called the parameter set and U is a set called the universal set. In other words, a soft set over a set U is a parametrized family of subsets of a universal set U. For example, when E is the set of parameters $\{red, blue, green, 2-door, sedan, hatch\_back\}$ for some set of cars U in a Car Lot, for the parameter $e = red$ in E, the set of all red cars in U, for the parameter $e = blue$ in E, the set of all blue cars in U, ... define a soft set over U.

Now a soft group over a group G, is any soft set (F, E) over the underlying set of the group G where $F: E \to P(G)$ is a mapping such that for each e in E, $F(e)$ is a subgroup of G. Just as a soft set over a set U is a parametrized family of subsets of the set U, a soft group over a group G is a parametrized family of subgroups of the group G. In fact, more generally, the notion of soft object over a crisp object U is a beautiful natural generalization of the crisp sub object, as the collection of all *non-empty* soft objects over U with a given parameter set over a given crisp object is in one-one correspondence with the collection of all crisp sub objects of the given object if the parameter set is a singleton. Notice that the empty soft object is the unique one with the parameter set, the empty set as, for a soft object (F, E) over an object U, E is empty implies F is empty.

Although soft sets are existing since their inception in 1999, interestingly soft maps didn't appear until 2011.

Noting that every soft set requires a universal set U, a parameter set E and a map $F: E \to P(U)$ and any three of them determine a soft set, Murthy-Maheswari[9] generalized the notion of soft set to that of a Generalized soft set as a triplet $(A, F, P(U))$ and the notion of soft map to that of a Generalized soft map, $(f, \phi): (A, F, P(U)) \to (B, G, P(V))$ where $f: A \to B$ is any map and $\phi: P(U) \to P(V)$ is any complete homomorphism, deriving several of their mapping properties from the generalized fuzzy sets and generalized fuzzy maps developed in Murthy[8] of 1997.

Now, Murthy-Gouthami[12] used the above notion of generalized soft (sub) set and introduced the notions of generalized soft group, generalized soft (normal) subgroup, generalized soft quotient group etc., generalizing the corresponding notions of a soft group over a group and showed that several of the crisp theoretic results naturally extended to these new objects too.

Now our aim in this paper is to introduce the notions of generalized soft homomorphism of generalized soft groups, generalized soft (inverse) image of generalized soft (normal) subgroup under generalized soft homomorphism, generalized soft kernel of a generalized soft homomorphism etc., generalizing the corresponding existing notions for a soft group over a group and show that several of the crisp theoretic results naturally extend to these new objects too.

According to[1] for any pair of soft sets (U, A) and (V, B), (i) a *soft map* is any pair (u, p), where $u: U \to V$ and $p: A \to B$ are mappings. (ii) for any soft set (W, C) in (U, A), the *soft image* of (W, C) under (u, p), denoted by $(u, p)(W, C)$, is defined by the soft set (Z, D), where $D = pC$ and $Zd = \cup uW(p^{-1}d \cap C) = u(\cup W(p^{-1}d \cap C))$. (iii) for any soft set (Z, D) in (V, B), the *soft inverse image* of (Z, D) under (u, p), denoted by $(u, p)^{-1}(Z, D)$, is defined by the soft set (W, C), where $C = p^{-1}D$ and $Wc = u^{-1}Zpc$.

According to [15] a soft mapping is defined as follows: let $S(X, A)$ and $S(Y, B)$ be the families of all soft sets over X and Y respectively. (iv) The mapping $f_\psi: S(X, A) \to S(Y, B)$ is called a soft mapping from X to Y, where $f: X \to Y$ and $\psi: A \to B$ are two mappings. Also (v) the *image* of a soft set $(F, A) \in S(X, A)$ under the mapping f_ψ is denoted by $f_\psi[(F, A)] = (f_\psi(F), B)$, and is defined by

$$[f_\psi(F)](\beta) = \begin{cases} \cup_{\alpha\in\psi^{-1}(\beta)} [f[F(\alpha)]] & \text{if } \psi^{-1}(\beta) \neq \phi \\ \phi & \text{otherwise, forall } \beta \in B \end{cases}$$

(vi) the *inverse image* of a soft set $(G, A) \in S(Y, B)$ under the mapping f_ψ is denoted by $f_\psi^{-1}[(G, B)] = (f_\psi^{-1}(G), A)$, and is defined by $[f_\psi^{-1}(G)](\alpha) = f^{-1}[G[\psi(\alpha)]]$, for all $\alpha \in A$. (iii) The soft mapping f_ψ is called injective (surjective) if f and ψ are both injective (surjective). (vii) The soft mapping f_ψ is identity soft mapping, if f and ψ are both classical identity mappings. Further, they showed that (inverse) image of a (fuzzy) soft (normal) subgroup is a (fuzzy) soft (normal) subgroup.

On the other hand, Murthy-Maheswari[9] generalized the notions of soft set over a universal set to that of a generalized soft set, introducing generalized soft subset and soft map between soft sets to that of a generalized soft map between generalized soft sets possibly with different universal sets, introducing generalized (inverse) image for generalized soft subsets. Later on we will see that the above definitions are special cases of generalized (inverse) image under generalized soft map.

II. PRELIMINARIES

In what follows we recall some basic definitions in the theory of Sets, Groups, Soft Sets, Soft maps, Generalized soft sets, s-map, and s-(inverse) images which are used in the main results.

(A) **Sets and Maps:** In order to make the document more self contained we begin with recalling some elementary notions and results in the theory of sets and groups: For any map $f: X \to Y$, (1) $f^{-1}fX = X$ (2) for all a, b in X, $a \sim b$ iff $fa = fb$ is an equivalence relation on X with eqivalence classes $(f^{-1}fa)_{a\in X}$, also called *kernel classes* (3) for any subsets $A \subseteq X$, $B \subseteq Y$, $fA \subseteq B$ iff $A \subseteq f^{-1}B$ (4) for any subset A of X, $A \subseteq f^{-1}fA$ (5) for any subset B of Y, $ff^{-1}B \subseteq B$ (6) for any pair of subsets A, C of X, $A \subseteq C$ implies $fA \subseteq fC$ (7) for any pair of subsets B, D of Y, $B \subseteq D$ implies $f^{-1}B \subseteq f^{-1}D$ (8) f is surjective iff $B = ff^{-1}B$ for any subset B of Y.

(B) **Power Sets and Power Maps:** (1) Since for each u in U, u can be identified with $\{u\}$ in $P(U)$, U can be naturally regarded as a subset of $P(U)$. Hence, we do *not* make any distinction between u and $\{u\}$ and write u in place of $\{u\}$. Consequently, any $F: P(U) \to P(V)$ naturally defines the *restricted* function $F|U: U \to P(V)$. Further, capital letters like A, B, C, D indicate elements of power set. Consequently, FA is a member of $P(V)$ and $F^{-1}B$ is the subset $\{A/FA = B\}$ of $P(U)$. (2) for any map $\phi: U \to V$, there is a natural extension map $P(\phi): P(U) \to P(V)$ defined by $P(\phi)(A) = \phi(A)$ for each *element* A in $P(U)$ where $\phi(A)$ is *the image of the subset A contained in U under* ϕ. (3) Now F is said to be *preserving* iff $F|U$ maps U into V.
Here onwards we denote empty set by ∅.

Notice that $P(F|U)$ need *not* equal F on $P(U)$, as shown in the following example:
Example 1: Let U=$\{x, y\}$, V=$\{a, b\}$ and $F: P(U) \to P(V)$ be given by $F = \{(x, a), (y, a), (\{x, y\}, b),$ (∅,∅)$\}$. Then $F|U: U \to V$ is $\{(x, a), (y, a)\}$, $P(F|U)(\{x, y\}) = \{a\} \neq \{b\} = F(\{x, y\})$.
(4) Hence $F: P(U) \to P(V)$ is said to be *extended* iff $F = P(F|U)$. In other words F is an extension of $F|U$. In this case, most of the times the base map $F|U$ is denoted simply by F_0.
(5) Observe that (i) if F extends two base maps ϕ and ψ then $\phi = \psi$ because for any u in U, $\phi(u) = P(\phi)(\{u\}) = F(u) = P(\psi)(\{u\}) = \psi(u)$ or $\phi = \psi$. (ii) even if $F: P(U) \to P(V)$ is preserving, as shown in the above example, F need *not* equal $P(F|U)$.

(6) Further, for all $F: P(U) \to P(V)$ such that F is extended,
(i) for all $A \subseteq U$, $P(F|U)(A) = (F|U)(A) = F_0(A)$.
(ii) F maps the element $\emptyset$ in $P(U)$ onto the element $\emptyset$ in $P(V)$ or $F(\emptyset) = \emptyset$ and the inverse image of the element $\emptyset$ in $P(V)$ is the element $\emptyset$ in $P(U)$ or $F^{-1}\{\emptyset\} = \{\emptyset\}$. In other words F preserves and reflects empty subsets whenever F is extended
(iii) for all $b \in V$ such that $F_0^{-1}b = \emptyset$, we have $F^{-1}B = \emptyset$ for all $B \in P(V)$ such that $b \in B$, as $A \in F^{-1}B$ implies $FA = B$; $b \in B = FA = F_0A$ implies there is $a \in A$ such that $F_0a = b$ or $a \in F_0^{-1}B$, contradicting $F_0^{-1}b$ is empty. In particular, if for all $b \in B$ $F_0^{-1}b = \emptyset$ then $F^{-1}B = \emptyset$.
(iv)(a) Always, for all $B \in P(V)$, $\cup F^{-1}B \subseteq F_0^{-1}B$ as, if $F^{-1}B$ is empty then the left hand side is empty giving the containment and if $F^{-1}B$ is non-empty then for all $A \in F^{-1}B$, $F_0A = FA = B$ which implies $F_0A = B$ which in turn implies $A \subseteq F_0^{-1}F_0A = F_0^{-1}B$ implying $\cup F^{-1}B \subseteq F_0^{-1}B$ as required.

The following Example shows that the above containment can be *proper*:
Example 2: Let U=$\{x, y, z\}$, V=$\{a, b, c\}$ and $F: P(U) \to P(V)$ be given by F= $\{(x, a), (y, b), (z, b), (\{x, y\}, \{a, b\}), (\{x, z\}, \{a, b\}), (\{y, z\}, b), (U, \{a, b\})\}$. Then $F_0: U \to V$ is $\{(x, a), (y, b), (z, b)\}$.
Let $B = \{b, c\}$. Then $F^{-1}B = \emptyset$, $\cup F^{-1}B = \emptyset$, $F_0^{-1}B = \{y, z\}$ implying $\cup F^{-1}B = \emptyset \subsetneq \{y, z\} = F_0^{-1}B$.
(b) On the other hand, for all $B \in P(V)$ such that $F^{-1}B \neq \emptyset$, we have $F_0^{-1}B \subseteq \cup F^{-1}B$. If $B = \emptyset$ then we have equality and so let $B \neq \emptyset$. Also, if $F_0^{-1}B = \emptyset$ then also we are done and so let $F_0^{-1}B \neq \emptyset$.
Now let $B_0 = \{b \in B / F_0^{-1}b \neq \emptyset\}$. If $B_0 = \emptyset$ then for all $b \in B$ $F_0^{-1}b = \emptyset$ then $F^{-1}B = \emptyset$, by 6(iii) which is not the case. So, $B_0 \neq \emptyset$.
Now for all $b \in B_0$, $\emptyset \neq F_0^{-1}b \subseteq F^{-1}b$ or $F^{-1}b \neq \emptyset$ and so $FF^{-1}b = \{b\}$. Let $A_0 = \cup_{b \in B_0} F^{-1}b$. Then $FA_0 = F(\cup_{b \in B_0} F^{-1}b) = \cup_{b \in B_0} FF^{-1}b = \cup_{b \in B_0} b = B_0 \subseteq B$ implies $FA_0 = B_0$.
Now $F^{-1}B \neq \emptyset$ implies for some $A_1 \in P(U)$ $FA_1 = B$. Let $A = A_0 \cup A_1$. Then $FA = FA_0 \cup FA_1 = B_0 \cup B = B$.
Now we show that $F_0^{-1}B \subseteq \cup F^{-1}B$. Let $a_0 \in F_0^{-1}B$. Then $F_0a_0 = b_0 \in B$ which implies $a_0 \in F_0^{-1}b_0 = F^{-1}b_0 \cap U \subseteq F^{-1}b_0$ which in turn implies $F_0^{-1}b_0 \neq \emptyset$ or $b_0 \in B_0$ and $a_0 \in F^{-1}b_0 \subseteq A_0 \subseteq A \subseteq \cup_{FA=B} A = \cup F^{-1}B$ or $F_0^{-1}B \subseteq \cup F^{-1}B$.
(c) Clearly, from (a) and (b) above it follows that for all $B \in P(V)$, if $F^{-1}B \neq \emptyset$ then we have $\cup F^{-1}B = F_0^{-1}B$ and if $F^{-1}B = \emptyset$ then we have $\cup F^{-1}B = \emptyset \subseteq F_0^{-1}B$ and in particular if $B = \emptyset$ then $\cup F^{-1} \emptyset = \emptyset = F_0^{-1} \emptyset$.

Example 3: Let $X = \{x, y, z, t\}$, $Y = \{a, b, c\}$.Then $P(X) = \{\phi, x, y, z, t, \{x, y\}, \{x, z\}, \{x, t\}, \{y, z\}, \{y, t\}, \{z, t\}, \{x, y, t\}, \{z, t, x\}, \{y, z, t\}, \{x, y, z\}, X\}$ and $P(Y) = \{\phi, a, b, c, \{a, b\}, \{b, c\}, \{c, a\}, Y\}$. Now let $F = \{(\emptyset, \emptyset), (x, a), (y, a), (z, c), (t, b), (\{z, t\}, \{b, c\}), (\{y, t\}, \{a, b\}), (\{y, z\}, \{a, c\}), (\{x, t\}, \{a, b\}), (\{x, z\}, \{a, c\}), (\{x, y\}, \{a\}), (\{x, y, t\}, \{a, b\}), (\{x, y, z\}, \{a, c\}), (\{y, z, t\}, Y), (\{z, x, t\}, Y), (X, Y)\}$.
Then (i) $F_0: X \to Y$ is $\{\{(x, a)\}, \{(y, a)\}, \{(z, c)\}, \{(t, b)\}\}$ (ii) for any element A in PX, $F(A) = F_0(A)$ where $F_0(A)$ is the image of the subset $A \subseteq X$ in Y under F_0 (iii) $F^{-1} \emptyset = \{\emptyset\}$, $F^{-1}Y = \{X, \{z, x, t\}, \{y, z, t\}\}$, $F^{-1}\{c\} = \{z\}$, $F^{-1}\{b\} = \{t\}$, $F^{-1}\{a\} = \{x, y\}$, $F^{-1}\{b, c\} = \{z, t\}$, $F^{-1}\{a, b\} = \{\{x, t\}, \{y, t\}, \{x, y, t\}\}$, $F^{-1}\{a, c\} = \{\{y, z\}, \{x, z\}, \{x, y, z\}\}$. (iv) Observe that, $F_0^{-1}Y = X$, $F_0^{-1}\phi = \phi$, $F_0^{-1}\{a, b\} = \{x, y, t\}$, $F_0^{-1}\{b, c\} = \{z, t\}$, $F_0^{-1}\{a, c\} = \{x, y, z\}$ and $F_0^{-1} \emptyset = \emptyset = \cup F^{-1} \emptyset$. Therefore for all B in $P(Y)$ such that $F^{-1}B \neq \emptyset$, $\cup F^{-1}Z = \cup_{FA=Z} A = F_0^{-1}Z$.

In what follows we show that the statements in (6) are *not* necessarily true if $F: P(U) \to P(V)$ does *not* extend $F_0 = F|U: U \to V$
First observe that even when $F: P(U) \to P(V)$ extends $F_0 = F|U: U \to V$, since F maps the element $\emptyset$ in $P(U)$ onto the element $\emptyset$ in $P(V)$ or $F(\emptyset) = \emptyset$, clearly, $\emptyset \in F^{-1}\{\emptyset\} \subseteq \cup F^{-1}\{\emptyset\}$ where as $F_0^{-1} \emptyset = \emptyset$ the empty set or (6) is *not* true for $\emptyset = B \subseteq V$.

Next as can be seen in Example 1 for an $F: P(U) \to P(V)$ even if $F|U$ maps U into V, F need *not* extend $F|U$.Even if $F: P(U) \to P(V)$ is such that $F_0 = F|U$ maps U into V, when F does *not* extend F_0, F need *not* map the element $\emptyset$ in $P(U)$ onto the element $\emptyset$ in $P(V)$ or $F(\emptyset) \neq \emptyset$ and/or the inverse image of the element $\emptyset$ in $P(V)$ need *not* be the singleton set consisting of the element $\emptyset$ in $P(U)$ or $F^{-1}(\{\emptyset\}) \neq \{\emptyset\}$ as shown in the following example:

Example 4: Let U=$\{x, y\}$, V=$\{a, b\}$ and $F: P(U) \to P(V)$ be given by F= $\{(x, a), (y, b), (\{x, y\}, \emptyset), (\emptyset, \{a, b\})\}$. Then $F|U: U \to V$ is $\{(x, a), (y, b)\}$, F does *not* extend $F|U$ because $F(\{x, y\}) = \emptyset \neq \{a, b\} = F_0(\{x, y\})$.
Clearly, $F(\emptyset) = \{a, b\} \neq \emptyset$ and $F^{-1}(\{\emptyset\}) = \{\{x, y\}\} \neq \{\emptyset\}$.
(d) For any map $F: P(U) \to P(V)$ such that F is extended, the following are true:
1. F_0 is onto iff F is onto as: for any $B \in P(V)$ and for any $b \in B$, $b = F_0a$ for some $a \in U$ or $F_0^{-1}b \neq \emptyset$; $A = \cup_{b \in B} F_0^{-1}b$ implies $F(A) = F_0A = F_0(\cup_{b \in B} F_0^{-1}b) = \cup_{b \in B} F_0F_0^{-1}b = B$ or F is onto;

F_0 is *not* onto implies there is a b_0 in V such that $F_0^{-1}b_0$ = ⍰ which by 6(iii) above implies for all super sets B in $P(V)$ with b in B have $F^{-1}B$ = ⍰, contradicting F being onto.
2. F is monotonic as, $A \subseteq B \subseteq U$ implies $F(A) = F_0(A) \subseteq F_0(B) = F(B)$.
3. whenever F is onto $C \subseteq D \subseteq V$ implies $\cup F^{-1}C \subseteq \cup F^{-1}D$ as, $\cup F^{-1}C = F_0^{-1}C \subseteq F_0^{-1}D = \cup F^{-1}D$.
In what follows we show that the above statement is not true if F is *not* onto:
In Example 2 above let $C = \{b\}$, $D = \{b, c\}$. Then $C \subseteq D$ but $\cup F^{-1}C = \{y, z\} \not\subseteq$ ⍰ $= \cup F^{-1}D$.

(C) **Groups:** (i) In any group G, arbitrary intersection of (normal) subgroups is a (normal) subgroup; for any pair of (normal) subgroups A, B of G, A is a (normal) subgroup of B iff A is a subset of B. (ii) The following are true for any group homomorphism $\phi: G \to H$: $Ker\phi \subseteq C$ and C is a subgroup of G imply $\phi^{-1}\phi C = C$; E is a subgroup of G and A is a (normal) subgroup of C imply $E \cap A$ is a (normal) subgroup of $E \cap C$; A is a (normal) subgroup of C implies ϕA is a (normal) subgroup of ϕC; B is a (normal) subgroup of D implies $\phi^{-1}B$ is a (normal) subgroup of $\phi^{-1}D$; A is normal subgroup of G implies A is a (normal) subgroup of $\phi^{-1}\phi A$ and $\phi^{-1}\phi A$ is normal subgroup of G; B is subgroup of H implies $\phi\phi^{-1}B$ is a subgroup of B (however, if B is a normal subgroup of H, as can be seen in $i: H_2 \to A_4$ the inclusion homomorphism, $B = A_4$, i $i^{-1}B = H_2$ is *not* normal in B.) and but if B is (normal) subgroup of H, $\phi\phi^{-1}B$ is (normal) subgroup of ϕG.

(D) **Soft Sets** In what follows we recall the following basic definitions from the Soft Set Theory which are used in due course: (1) [3] Let U be a universal set, $P(U)$ be the power set of U and E be a set of parameters. A pair (F, E) is called a *soft set* over U iff $F: E \to P(U)$ is a mapping defined by, for each $e \in E$ $F(e)$ is a subset of U. In other words, a soft set over U is a parametrized family of subsets of U. (2) For any soft set (F, E) over U, we let (F, E_r) be the *associated regular soft set* where $E_r = \{e \in E / Fe \neq \phi\}$. Clearly, a soft set (F, E) over U is non-null (cf.(7) below) iff $E_r \neq \phi$.

Notice that a collective presentation of all the notions, soft sets and gs-sets raised some serious notational conflicts and to fix the same Murthy-Maheswari[11] deviated from the above notation for a soft set and adapted the following notation for convenience as follows:

Let U be a universal set. A typical *soft set* over U is an ordered pair (San Serif) $S = (\sigma_S, S)$, where S is a set of *parameters*, called the *underlying parameter set* for S, $P(U)$ is the power set of U and $\sigma_S: S \to P(U)$ is a map, called the *underlying set valued map* for S. Some times σ_S is also called the *soft structure* on S.

(3) [7] The *empty* soft set over U is a soft set with the empty parameter set, denoted by $\Phi = (\sigma_\phi, \phi)$. Clearly, it is unique. (4) [6] A soft set S over U is said to be a *whole* soft set, denoted by U_S, iff $\sigma_S s = U$ for all $s \in S$. (5) [7] A soft set S over U is said to be a *null* soft set, denoted by Φ_S, iff $\sigma_S s = \phi$, the empty set, for all $s \in S$. Notice that $\Phi_\phi = \phi$, the empty soft (sub) set.
For any pair of soft sets A, B over U,
(6) [4] A is a *soft subset* of B, denoted by $A \subseteq B$, iff (i) $A \subseteq B$ (ii) $\sigma_A a \subseteq \sigma_B a$ for all $a \in A$. The set of *all soft subsets* of B is denoted by $\mathcal{S}_U(B)$
(7) The following are easy to see:
(i) Always the empty soft set Φ is a soft subset of every soft set A. (ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$ iff $A = B$ and $\sigma_A a = \sigma_B a$ for all $a \in A$.
(8) For any family of soft subsets $(A_i)_{i \in I}$ of S,
(i) the *soft union* of $(A_i)_{i \in I}$, denoted by $\cup_{i \in I} A_i$, is defined by the soft set A, where (i) $A = \cup_{i \in I} A_i$ (ii) $\sigma_A a = \cup_{i \in I_a} \sigma_{A_i} a$, where $I_a = \{i \in I / a \in A_i\}$, for all $a \in A$
(ii) the *soft intersection* of $(A_i)_{i \in I}$, denoted by $\cap_{i \in I} A_i$, is defined by the soft set A, where (i) $A = \cap_{i \in I} A_i$ (ii) $\sigma_A a = \cap_{i \in I} \sigma_{A_i} a$ for all $a \in A$.
Notice that $\cap_{i \in I} A_i$ can become empty soft set.

(E) **Soft Groups, Soft Group homomorphisms:** In this section we first recall the existing notions of a soft group, soft (normal) subgroup, soft group homomorphism etc.. (1)[5] if (F, A) is a soft set over a group G, then (F, A) is said to be a *soft group* over G if and only if $F(x) \leq G$ for all $x \in A$. (2)[5] Let (F, A) and (H, K) be two soft groups over G. Then (H, K) is a *soft subgroup* of (F, A), written as $(H, K) \leq (F, A)$, if $K \subseteq A$, $H(x) \leq F(x)$ for all $x \in K$. (3)[5] Let (F, A) be a soft group over G and (H, B) be a soft subgroup of (F, A). Then we say that (H, B) is a *normal soft subgroup* of (F, A), written $(H, B) \trianglelefteq (F, A)$, if $H(x)$ is a normal subgroup of $F(x)$ i.e., $H(x) \trianglelefteq F(x)$, for all $x \in B$.
(4)[2] if G is a group and (F, A) is a non-null soft set over G, then (F, A) is called a *normalistic soft group* over G if $F(x)$ is a normal subgroup of G for all $x \in Supp(F, A)$.

(5)[5], let (F,A) and (G,B) be two groups over G and K respectively, and let $f: G \to K$ and $g: A \to B$ be two functions. Then we say that (f,g) is a *soft homomorphism*, and that (F,A) is soft homomorphic to (H,B). The latter is written as $(F,A) \sim (H,B)$, if the following are satisfied: f is a homomorphism from G onto K, g is a mapping from A onto B, and $f(F(x)) = H(g(x))$ for all $x \in A$. In this definition, if f is an isomorphism from G to K and g is a one-to-one mapping from A onto B. then we say that (f,g) is a *soft isomorphism* and that (F,A) is soft isomorphic to (G,B). The latter is denoted by $(F,A) \backsimeq (H,B)$.

(F) **Generalized soft sets** In this section we recall the notions of generalized soft set, gs-set or s-set for short, gs-subset or s-subset, gs-union or s-union, gs-intersection or s-intersection etcetera from Murthy-Maheswari[9].
From now on, the script letters $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ etc. denote s-sets and/or their subsets and any such script letter $\mathcal{Q}$ stands for the triplet $(Q,\overline{Q},P(U_Q))$.
(1) A *generalized soft set* or an *s-set* in short, is any triplet $\mathcal{A}$, where A is the *underlying set of parameters for* U_A *or parameter set* in short, $P(U_A)$ is *the complete lattice of all subsets of* U_A *parametrized under* $\overline{A}$ *with parameters from* A and $\overline{A}: A \to P(U_A)$ is the *underlying parametrizing map for* U_A.
(2) For any soft set (F,A) over a universal set U with the parameter set A, the *associated s-set for* (F,A), is defined by the s-set $(A,F,P(U))$, where A is the underlying parameter set, $P(U)$ is the power set of all subsets of U and $F: A \to P(U)$ is the parametrizing map.
(3) The s-set $\mathcal{A}$, where A = , the empty set with *no* elements, $P(U_A)$ = {}, and $\overline{A}$ = , the empty map, is called the *empty s-set* and is denoted by .
(4) An s-set $\mathcal{A}$ is said to be a *whole s-set* iff the parametrizing map $\overline{A}: A \to P(U_A)$ is defined by $\overline{A}a = U_A$ for all $a \in A$.
(5) An s-set $\mathcal{A}$ is said to be a *null s-set* iff the parametrizing map $\overline{A}: A \to P(U_A)$ is defined by $\overline{A}a$ = , the empty set, for all $a \in A$.
(6) For any pair of s-sets $\mathcal{S}$ and $\mathcal{A}$, $\mathcal{S}$ is an *s-subset* of $\mathcal{A}$, denoted by $\mathcal{S} \subseteq \mathcal{A}$, iff (i) $S \subseteq A$ (ii) $U_S \subseteq U_A$ or equivalently $P(U_S)$ is a complete ideal of $P(U_A)$ and (iii) $\overline{S}a \subseteq \overline{A}a$ for all $a \in S$. Clearly, for any pair of s-subsets $\mathcal{S},\mathcal{T}$, $\mathcal{S} = \mathcal{T}$ iff $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{T} \subseteq \mathcal{S}$ iff $S = T$, $U_S = U_T$ or $P(U_S) = P(U_T)$ and $\overline{S} = \overline{T}$.
An s-subset $\mathcal{S}$ is *degenerated* iff S = and $\overline{S}$ = , the empty map. Clearly, the empty s-set is degenerated. Note that degenerated s-subset is *not* unique.
The set of all s-subsets of the s-set $\mathcal{B}$ is denoted by $\mathcal{S}(\mathcal{B})$.
Clearly, (i) the null s-set and the empty s-set are s-subsets of $\mathcal{A}$, but *not* necessarily the whole s-set. (ii) for any pair of soft sets (F,A) and (G,B) over U, (G,B) is a soft subset of (F,A) iff $(B,G,P(U))$ is an s-subset of $(A,F,P(U))$.
(7) The following are easy to see:
(i) Always the empty s-set is an s-subset of every s-set $\mathcal{A}$.
(ii) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$ iff $A = B$, $U_A = U_B$ and $\overline{A} = \overline{B}$.
(8) For any family of s-subsets $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{B}$,
(1) the *s-union* of $(\mathcal{A}_i)_{i\in I}$, denoted by $\cup_{i\in I} \mathcal{A}_i$, is defined by the s-set $\mathcal{A}$, where
(i) $A = \cup_{i\in I} A_i$ is the usual set union of the collection $(A_i)_{i\in I}$ of subsets of B.
(ii) $P(U_A) = \vee_{i\in I} P(U_{A_i}) = P(\cup_{i\in I} U_{A_i})$, where $\vee_{i\in I} P(U_{A_i})$ is the complete ideal generated by $\cup_{i\in I} P(U_{A_i})$ in $P(U_A)$ which is the same as $P(\cup_{i\in I} U_{A_i})$.
(iii) $\overline{A}: A \to P(U_A)$ is defined by $\overline{A}a = \cup_{i\in I_a} \overline{A}_i a$, where $I_a = \{i \in I | a \in A_i\}$
(2) the *s-intersection* of $(\mathcal{A}_i)_{i\in I}$, denoted by $\cap_{i\in I} (\mathcal{A}_i)$, is defined by the s-set $\mathcal{A}$, where
(i) $A = \cap_{i\in I} A_i$ is the usual set intersection of the collection $(A_i)_{i\in I}$ of subsets of B
(ii) $P(U_A) = \cap_{i\in I} P(U_{A_i}) = P(\cap_{i\in I} U_{A_i})$ is the usual intersection of the complete ideals of $P(U_{A_i})_{i\in I}$ in $P(U_A)$
(iii) $\overline{A}: A \to P(U_A)$ is defined by $\overline{A}a = \cap_{i\in I} \overline{A}_i a$.

(G) **gs-map, Images and Inverse Images of gs-Subsets under gs-maps**
In this section we recall the notions of gs-map, increasing gs-map, decreasing gs-map and preserving gs-map, gs-image and gs-inverse image of a gs-subset or s-subset under a gs-map from [9].
(1) For any pair of s-sets $\mathcal{A}$ and $\mathcal{B}$, a gs-*map* $\mathcal{F}$ is any pair (f,F), where $f: A \to B$ is a map and $F: P(U_A) \to P(U_B)$ is a complete homomorphism and is denoted by $\mathcal{F}: \mathcal{A} \to \mathcal{B}$.
(2) For any gs-map $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, $\mathcal{F}$ is
(i) *increasing*, denoted by $(f,F)_i$ iff $\overline{B}fa \supseteq F\overline{A}a$ for all a in A.
(ii) *decreasing*, denoted by $(f,F)_d$ iff $\overline{B}fa \subseteq F\overline{A}a$ for all a in A.
(iii) *preserving*, denoted by $(f,F)_p$ iff $\overline{B}fa = F\overline{A}a$ for all a in A..

(3) For any gs-map $\mathcal{F}: \mathcal{A} \to \mathcal{B}$,
(i) for any gs-subset $\mathcal{C}$ of $\mathcal{A}$, the *gs-image* of $\mathcal{C}$, denoted by $\mathcal{F}\mathcal{C}$, is defined by the gs-set $\mathcal{D}$, where (i) $D = f(C)$ (ii) $U_D = F(U_C)$ or $P(U_D) = (F(P(U_C)))_{P(U_B)} = P(F(U_C)) = (F(U_C))_{P(U_B)}$ (iii) $\overline{D}: D \to P(U_D)$ is given by $\overline{D}d = \overline{B}d \cap\cup F\overline{C}(f^{-1}d \cap C)$ for all $d \in D$
(ii) for any gs-subset $\mathcal{D}$ of $\mathcal{B}$, the *gs-inverse image* of $\mathcal{D}$, denoted by $\mathcal{F}^{-1}\mathcal{D}$, is defined by the gs-set $\mathcal{C}$, where (i) $C = f^{-1}(D)$ (ii) $U_C = F^{-1}(U_D)$ or $P(U_C) = F^{-1}(P(U_D)) = P(F^{-1}(U_D))$ (iii) $\overline{C}: C \to P(U_C)$ is given by $\overline{C}c = \overline{A}c \cap\cup F^{-1}\overline{D}fc$ for all $c \in C$.

Notice that the notions of gs-(sub) set, gs-map and gs-(inverse) image are actually special cases of the so called f-(sub) set, f-map and f-(inverse) image which are introduced in 1997 by Murthy[8] as follows:
(4) An *f-set* is a triplet (A, X, L), where A is the *underlying set* of/for (A, X, L), L is the *underlying complete lattice of truth values* of/for (A, X, L) and $X: A \to L$ is the *underlying fuzzy map* of/for (A, X, L). In an f-set (A, X, L), A, L and X are uniquely determined. Any f-set is a *boolean f-set* iff L is a boolean algebra.
For any pair of f-sets (A, X, L) and (B, Y, M), (5) (A, X, L) is an *f-subset* of (B, Y, M), denoted by $(A, X, L) \subseteq (B, Y, M)$, iff (i) A is a subset of B (ii) L is a complete ideal of M (iii) $X \leq Y|A$.
For any family of f-subsets $(A_i, X_i, L_i)_{i\in I}$ of (B, Y, M), (6) the *f-union* of $(A_i, X_i, L_i)_{i\in I}$, denoted by $\cup_{i\in I} (A_i, X_i, L_i)$, is defined by the f-set (A, X, L), where (i) $A = \cup_{i\in I} A_i$ is the usual set union of the collection $(A_i)_{i\in I}$ of sets (ii) $L = \vee_{i\in I} L_i$, where $\vee_{i\in I} L_i$ is the complete ideal generated by $\cup_{i\in I} L_i$ in L (iii) $X: A \to L$ is defined by $Xa = \vee_{i\in I_a} X_i a$, where $I_a = \{i \in I | a \in A_i\}$. (7) the *f-intersection* of $(A_i, X_i, L_i)_{i\in I}$, denoted by $\cap_{i\in I} (A_i, X_i, L_i)$, is defined by the f-set (A, X, L), where (i) $A = \cap_{i\in I} A_i$ is the usual set intersection of the collection $(A_i)_{i\in I}$ of sets (ii) $L = \cap_{i\in I} L_i$ is the usual intersection of the complete ideals $(L_i)_{i\in I}$ in L (iii) $X: A \to L$ is defined by $Xa = \wedge_{i\in I} X_i a$.
(8) For any pair of f-sets (A, X, L) and (B, Y, M), the pair (f, ϕ), where $f: A \to B$ is a map and $\phi: L \to M$ is a complete homomorphism, is said to be an *f-map* and is denoted by $(f, \phi): (A, X, L) \to (B, Y, M)$.
An f-map $(f, \phi): (A, X, L) \to (B, Y, M)$ is a *boolean f-map* iff both (A, X, L) and (B, Y, M) are boolean f-sets and $\phi: L \to M$ is a complete homomorphism of complete lattices.
For any f-map $(f, \phi): (A, X, L) \to (B, Y, M)$, (9) for any f-subset (C, W, K) of (A, X, L), the *f-image* of (C, W, K), denoted by $(f, \phi)(C, W, K)$, is defined by the f-set (D, Z, N), where (i) $D = fC$ (ii) $N = (\phi K)_M$ (iii) $Z: D \to N$ is given by $Zd = Yd \wedge\vee \phi W(f^{-1}d \cap C)$ for all $d \in D$ (10) for any f-subset (D, Z, N) of (B, Y, M), the *f-inverse image* of (D, Z, N), denoted by $(f, \phi)^{-1}(D, Z, N)$, is defined by the f-set (C, W, K), where (i) $C = f^{-1}D$ (ii) $K = \phi^{-1}N$ (iii) $W: C \to K$ is given by $Wc = Xc \wedge\vee \phi^{-1}Zfc$ for all $c \in C$.

Clearly, a boolean f-set (A, X, L) is a gs-set iff L is the particular boolean algebra $P(U)$ for some universal set U and a boolean f-map $(f, \phi): (A, X, L) \to (B, Y, M)$ is a gs-map iff $\phi: L \to M$ is a complete homomorphism of complete lattices $L = P(U)$ for some U and $M = P(V)$ for some V.

Any power algebra is of the form $P(U)$ for some set U, in any power algebra $P(U)$, the largest element $1_{P(U)}$ is U and the least element $0_{P(U)}$ is ?, the empty set and some times we write fa, $f^{-1}b$, FA, $F^{-1}B$, PU instead of $f(a)$, $f^{-1}(b)$, $F(A)$, $F^{-1}(B)$, $P(U)$ etc. respectively.

III S-SETS, S-MAPS, (INVERSE) IMAGES OF S-SUBSETS

In this section we introduce the notions of s-maps between s-sets with possibly different universal sets, (inverse) images of s-subsets under s-maps.

The generalized soft set introduced in [9] is simply called s-set in this paper. However, the s-maps to be introduced in this paper will be different from the s-maps introduced in [9].
(a) For any pair of s-sets $\mathcal{A}$ and $\mathcal{B}$, an *s-map* is any pair (f, F), denoted by $\mathcal{F}$, where $f: A \to B$ and $F: P(U_A) \to P(U_B)$ is onto and extends $F|U_A$ or equivalently F=$P(F|U_A)$.
Note that quite often in all our examples $F|U_A$ will be denoted by F_0.

In what follows we give an Example to show that if F is *not* onto then some of the crucial properties like, for subsets $\mathcal{C}, \mathcal{D}$ of $\mathcal{B}$, $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{B}$ implies $\mathcal{F}^{-1}\mathcal{C} \subseteq \mathcal{F}^{-1}\mathcal{D}$; $\mathcal{F}^{-1}(\mathcal{C} \cup \mathcal{D}) = \mathcal{F}^{-1}\mathcal{C} \cup \mathcal{F}^{-1}\mathcal{D}$; $\mathcal{F}^{-1}(\mathcal{C} \cap \mathcal{D}) = \mathcal{F}^{-1}\mathcal{C} \cap \mathcal{F}^{-1}\mathcal{D}$ etc. do *not* hold and, as we know, without them nothing much can be done:
Example 5: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be given by $\mathcal{A} = (\{p\}, \{(p, \{x, y, z\}), P(\{x, y, z\})\})$, $\mathcal{B} = (\{q\}, \{(q, \{a, b, c\}), P(\{a, b, c\})\})$, $f = \{(p, q)\}$ and F=$\{(x, a), (y, b), (z, b), (\{x, y\}, \{a, b\}), (\{x, z\}, \{a, b\}), (\{y, z\}, b), (U, \{a, b\})\}$.
Then F_0= $\{(x, a), (y, b), (z, b)\}$.
(1) Let $\mathcal{C} = (\{q\}, \{(q, \{b\}), P(\{a, b, c\})\})$, $\mathcal{D} = (\{q\}, \{(q, \{b, c\}), P(\{a, b, c\})\})$.

Let $\mathcal{F}^{-1}\mathcal{C} = \mathcal{M}$. Then $M = f^{-1}C = \{p\}$, $U_M = F_0^{-1}U_C = \{x, y, z\}$ and $\overline{M}p = \overline{A}p \cap\cup F^{-1}\overline{C}fp = \{x, y, z\} \cap \{y, z\} = \{y, z\}$.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{N}$. Then $N = f^{-1}D = \{p\}$, $U_N = F_0^{-1}U_D = \{x, y, z\}$ and $\overline{N}p = \overline{A}p \cap\cup F^{-1}\overline{C}fp = \{x, y, z\} \cap □ = □$, implying $\mathcal{C} \subseteq \mathcal{D}$ but $\mathcal{F}^{-1}\mathcal{C} \not\subseteq \mathcal{F}^{-1}\mathcal{D}$.
(2) Let $\mathcal{C} = (\{q\}, \{(q, \{b\}), P(\{a, b, c\})\})$, $\mathcal{D} = (\{q\}, \{(q, \{c\}), P(\{a, b, c\})\})$. Then $\mathcal{C} \cup \mathcal{D} = (\{q\}, \{(q, \{b, c\}), P(\{a, b, c\})\})$, $\mathcal{F}^{-1}\mathcal{C} = (\{p\}, \{(p, \{y, z\}), P(\{x, y, z\})\})$, $\mathcal{F}^{-1}\mathcal{D} = (\{p\}, \{(p, □), P(\{x, y, z\})\})$. $\mathcal{F}^{-1}\mathcal{C} \cup \mathcal{F}^{-1}\mathcal{D} = (\{p\}, \{(p, \{y, z\}), P(\{x, y, z\})\})$ and $\mathcal{F}^{-1}(\mathcal{C} \cup \mathcal{D}) = (\{p\}, \{(p, □), P(\{x, y, z\})\})$, implying $\mathcal{F}^{-1}(\mathcal{C} \cup \mathcal{D}) \neq \mathcal{F}^{-1}\mathcal{C} \cup \mathcal{F}^{-1}\mathcal{D}$.
(3) Let $\mathcal{C}=(\{q\}, \{(q, \{b, c\}), P(\{a, b, c\})\})$, $\mathcal{D}=(\{q\}, \{(q, \{a, b\}), P(\{a, b, c\})\})$.Then $\mathcal{C} \cap \mathcal{D} = (\{q\}, \{(q, \{a, b\}), P(\{a, b, c\})\})$, $\mathcal{F}^{-1}\mathcal{C} = (\{p\}, \{(p, □), P(\{x, y, z\})\})$, $\mathcal{F}^{-1}\mathcal{D} = (\{p\}, \{(p, \{x, y, z\}), P(\{x, y, z\})\})$, $\mathcal{F}^{-1}\mathcal{C} \cap \mathcal{F}^{-1}\mathcal{D} = (\{p\}, \{(p, □), P(\{x, y, z\})\})$, $\mathcal{F}^{-1}(\mathcal{C} \cap \mathcal{D}) = (\{p\}, \{(p, \{x, y, z\}), P(\{x, y, z\})\})$, implying $\mathcal{F}^{-1}(\mathcal{C} \cap \mathcal{D}) \neq \mathcal{F}^{-1}\mathcal{C} \cap \mathcal{F}^{-1}\mathcal{D}$.

Here onwards all our s-maps are the ones defined as above.

Observe that whenever, $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an s-map, F is onto and extended, and so (1) for all $B \in P(U_B)$, $\cup F^{-1}B = (F|U)^{-1}B = F_0^{-1}B$. (2) for all $C \subseteq U_A$, the image of the element $C \in P(U_A)$ under F is the same as the image of the set C under F_0 or $FC = F_0C$ (3) Consequently, $\mathcal{F}$ is increasing iff $\overline{B}fa \supseteq F\overline{A}a$ for all a in A iff $\overline{B}fa \supseteq F_0\overline{A}a$ for all a in A, $\mathcal{F}$ is decreasing iff $\overline{B}fa \subseteq F\overline{A}a$ for all a in A iff $\overline{B}fa \subseteq F_0\overline{A}a$ for all a in A, $\mathcal{F}$ is preserving iff $\overline{B}fa = F\overline{A}a$ for all a in A iff $\overline{B}fa = F_0\overline{A}a$ for all a in A.
(b) Consequently, for any s-map $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ we define,
(i) $\mathcal{F}$ is *increasing*, denoted by $\mathcal{F}_i$ iff $\overline{B}fa \supseteq F_0\overline{A}a$ for all a in A
(ii) $\mathcal{F}$ is *decreasing*, denoted by $\mathcal{F}_d$ iff $\overline{B}fa \subseteq F_0\overline{A}a$ for all a in A
(iii) $\mathcal{F}$ is *preserving*, denoted by $\mathcal{F}_p$ iff $\overline{B}fa = F_0\overline{A}a$ for all a in A.

Observe that the s-map defined here in this paper is slightly different from the s-map defined in [9] (cf.(a) above) in the sense that there F is a complete homomorphism of complete lattices which is needed to extend several of the usual set-map properties proved in the same paper. Further, even the increasing, decreasing and preserving maps defined in this paper are slightly different from the corresponding ones defined in [9] (cf.(b) above), in the sense that there, F is a complete homomorphism of complete lattices but here F is an extended s-map.

For any s-map $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ (defined as in (a) above)
(c) for any s-subset $\mathcal{C}$ of $\mathcal{A}$, the *s-image* of $\mathcal{C}$ under $\mathcal{F}$, denoted by $\mathcal{F}\mathcal{C}$, is defined by the s-set $\mathcal{D}$, where (i) $D = fC$ (ii) $U_D = F_0U_C$ or $PU_D = (F_0PU_C)_{P(U_B)} = P(F_0U_C) = (F_0U_C)_{P(U_B)}$ (iii) $\overline{D}: D \to PU_D$ is given by $\overline{D}d = \overline{B}d \cap\cup F\overline{C}(f^{-1}d \cap C)$ for all $d \in D$
Observe that in the light of Remarks after Example 6 above, we get that $\overline{D}d = \overline{B}d \cap\cup_{c \in f^{-1}d \cap C} F\overline{C}c = \overline{B}d \cap \cup_{c \in f^{-1}d \cap C} F_0\overline{C}c$ for all $d \in D$.

Clearly, when $\mathcal{B}$ is a whole s-set, namely, $\overline{B}b = U_B$ our s-image of $\mathcal{C}$ under $\mathcal{F}$ reduces to the definitions of soft image of a soft set in [14] and image of a soft set in [13].

The following example shows that in the above without the term, $\overline{B}d$ the s-set $\mathcal{D}$ need *not* be an s-subset of $\mathcal{B}$.
Example 6: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, $\mathcal{D} \subseteq \mathcal{B}$ be given by: $\mathcal{A}=(\{a\}, \{(a, p)\}, \{□, \{p\}\}) = \mathcal{C}$, $\mathcal{B} = (\{b\}, \{(b, □)\}, \{□, \{q\}\})$, $f: A \to B$ is given by $f = \{(a, b)\}$ and $F_0 = \{(□, □), (p, q)\}$.
Let $\mathcal{D} = \mathcal{F}\mathcal{C}$. Then $D = fC = \{b\} = B$, $U_D = F_0U_C = F_0\{p\} = \{q\}$ and $\overline{D}b = \cup_{c \in f^{-1}b \cap C} F_0\overline{C}c = F_0\overline{C}a = F_0\{p\} = \{q\}$, implying $\mathcal{D} = (\{b\}, \{(b, q)\}, \{□, \{q\}\})$. Clearly, $\mathcal{D}$ is *not* an s-subset of $\mathcal{B}$ because $\overline{D}b = \{q\} \not\subseteq \overline{B}b = □$.
(d) for any s-subset $\mathcal{D}$ of $\mathcal{B}$, the *s-inverse image* of $\mathcal{D}$ under $\mathcal{F}$, denoted by $\mathcal{F}^{-1}\mathcal{D}$, is defined by the s-set $\mathcal{C}$, where (i) $C = f^{-1}(D)$ (ii) $U_C = F_0^{-1}(U_D)$ or $P(U_C) = F_0^{-1}(P(U_D)) = P(F_0^{-1}(U_D))$ (iii) $\overline{C}: C \to P(U_C)$ is given by $\overline{C}c = \overline{A}c \cap\cup F^{-1}\overline{D}fc$ for all $c \in C$.

Observe that in the light of Remarks after Example 6 above, we get that for all $c \in C$, $\overline{C}c = \overline{A}c \cap F_0^{-1}\overline{D}fc$.
Clearly, when $\mathcal{A}$ is a whole s-set, namely, $\overline{A}a = U_A$ our s-inverse image of $\mathcal{D}$ under $\mathcal{F}$ reduces to the definitions of soft inverse image of a soft set in [14] and inverse image of a soft set in [13].

The following example shows that in the above without the term, $\overline{A}c$ the s-set $\mathcal{C}$ need *not* be an s-subset of $\mathcal{A}$.

Example 7: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, $\mathcal{C} \subseteq \mathcal{A}$ be given by:$\mathcal{A}$=$(\{a\}, \{(a,□)\}, \{□, \{p\}\})$, $\mathcal{B}$ = $(\{b\}, \{(b,q)\}, \{□, \{q\}\})$ = $\mathcal{D}$, $f: A \to B$ is given by $f = \{(a,b)\}$ and $F_0 = \{(□,□)(p,q)\}$.
Let $\mathcal{C} = \mathcal{F}^{-1}\mathcal{D}$. Then $C = f^{-1}D = \{a\}$, $U_C = F_0^{-1}U_D = F_0^{-1}\{q\} = \{p\}$ and $\overline{C}a = F_0^{-1}\overline{D}fa = F_0^{-1}\overline{D}b = F_0^{-1}\{q\}$ = $\{p\}$, implying $\mathcal{C} = (\{a\}, \{(a,p)\}, \{□, \{p\}\})$. Clearly, $\mathcal{C}$ is *not* an s-subset of $\mathcal{A}$ because $\overline{C}a = \{p\} \not\subseteq \overline{A}a$ = □.
(e) For any s-map $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, an s-subset $\mathcal{C}$ of $\mathcal{A}$ is an $\mathcal{F}$*-constant* or *constant on each kernel class* iff $\overline{C}a = \overline{C}c$ for all $a \in f^{-1}fc$ for all $c \in C$.

Lemma 3.1 *For any s-map $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ and for any s-subset $\mathcal{D}$ of $\mathcal{B}$, $\mathcal{F}^{-1}\mathcal{D}$ is always an $\mathcal{F}$-constant subset of $\mathcal{A}$, whenever $\mathcal{A}$ is $\mathcal{F}$-constant.*
Proof: Since $\mathcal{A}$ is $\mathcal{F}$-constant, $\overline{A}a = \overline{A}c$ for all $a \in f^{-1}fc$ for all $c \in A$. Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{C}$. Then $C = f^{-1}D$, U_C = $F_0^{-1}U_D$ and $\overline{C}c = \overline{A}c \cap F_0^{-1}\overline{D}fc$ for all $c \in C$.
Let $c \in C$ and $a \in f^{-1}fc$. Then $fa = fc$ and since $\mathcal{A}$ is $\mathcal{F}$-constant $\overline{C}a = \overline{A}a \cap F_0^{-1}\overline{D}fa = \overline{A}c \cap F_0^{-1}\overline{D}fc = \overline{C}c$.

In what follows we show that if $\mathcal{A}$ is $\mathcal{F}$-constant then any s-subset $\mathcal{C}$ of $\mathcal{A}$ need *not* be $\mathcal{F}$-constant:
Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be given by: $\mathcal{A} = (\{a,b\}, \{(a,\mathbb{Z}_6),(b,\mathbb{Z}_6)\}, P(\mathbb{Z}_6))$, $\mathcal{B} = (\{c\}, \{(c,\mathbb{Z}_6)\}, P(\mathbb{Z}_6))$, f = $\{(a,c),(b,c)\}$, $F_0 = 1_{\mathbb{Z}_6}$ and $\mathcal{C} = (\{a,b\}, \{(a,Z_2),(b,Z_3)\}, P(\mathbb{Z}_6))$, where $Z_2 = \{\overline{0},\overline{3}\}$ and $Z_3 = \{\overline{0},\overline{2},\overline{4}\}$.
Then clearly, $\mathcal{A}$ is $\mathcal{F}$-constant, $\mathcal{C}$ is an s-subset of $\mathcal{A}$ but $\mathcal{C}$ is not an $\mathcal{F}$-constant s-subset.
Similarly, it is easy to show that s-superset of any $\mathcal{F}$-constant s-subset need *not* be $\mathcal{F}$-constant either.

IV S-GROUPS, S-HOMOMORPHISMS OF S-GROUPS

In this section we recall the notions of s-group, s-(normal) subgroup from Murthy-Gouthami[12].
Definitions and Statements 4.1 *(a) An s-set $\mathcal{G}$ is said to be an s-(normal) group iff (i) U_G is a group (ii) for all $g \in G$, $\overline{G}g$ is a (normal) subgroup of U_G.*
An s-group which is also a whole s-set is a *whole s-group*. Clearly, whole s-group and whole s-normal group are the same.
(b) For any s-group $\mathcal{G}$ and for any s-subset $\mathcal{A}$ of $\mathcal{G}$,
(1) $\mathcal{A}$ is an *s-subgroup* of $\mathcal{G}$ iff (i) $A \subseteq G$ (ii) U_A is a subgroup of U_G (iii) $\overline{A}g$ is a subgroup of $\overline{G}g$ for all $g \in A$.
An s-subgroup $\mathcal{A}$ is an *identity s-subgroup* of $\mathcal{G}$ iff $U_A = (e_{U_G})$ and $\overline{A}g = (e_{U_G})$ for all $g \in A$.
Clearly, (i) the s-group $\mathcal{G}$ itself and any identity s-subgroup of $\mathcal{G}$ are always s-subgroups of $\mathcal{G}$ but both the null s-subset and the empty s-subset of an s-group $\mathcal{G}$ are *not* s-subgroups of $\mathcal{G}$. (ii) a degenerated s-subset $\mathcal{A}$ of $\mathcal{G}$ is an s-subgroup iff U_A is a subgroup of U_G.
For any soft group (F,A) over a group U, the associated s-set $(A,F,P(U))$ of (F,A) (cf.2.e(2)) is an s-subgroup of the whole s-group $\mathcal{G}$ where $G = A$ and $U_G = U$, called the *associated s-subgroup* for (F,A).
Clearly, (i) for any pair of soft sets (F,A) and (G,B) over a group U, (G,B) is a soft subgroup of (F,A) iff $(B,G,P(U))$ is an s-subgroup of $(A,F,P(U))$. (ii) for any s-group $\mathcal{G}$ and for any c-total s-subset $\mathcal{A}$ of $\mathcal{G}$, $(A,\overline{A})$ is a soft subgroup of the soft group $(G,\overline{G})$ over U_G iff $\mathcal{A}$ is an s-subgroup of $\mathcal{G}$.
(2) $\mathcal{A}$ is an *s-normal subgroup* of $\mathcal{G}$ iff (i) $A \subseteq G$ (ii) U_A is a normal subgroup of U_G (iii) $\overline{A}g$ is a normal subgroup of $\overline{G}g$ for all $g \in A$.
Clearly, (i) every s-normal subgroup is an s-subgroup but *not* conversely (ii) the s-group itself and any identity s-subgroup are always s-normal subgroups of $\mathcal{G}$ but both the null s-subset and the empty s-subset of an s-group $\mathcal{G}$ are *not* s-normal subgroups of $\mathcal{G}$. (iii) a degenerated s-subset $\mathcal{A}$ of $\mathcal{G}$ is an s-normal subgroup iff U_A is a normal subgroup of U_G and a degenerated s-subset which is also an s-(normal) subgroup is called a *degenerated s-(normal) subgroup*.
(3) For any pair of s-(normal) subgroups $\mathcal{A}, \mathcal{B}$ of $\mathcal{G}$, $\mathcal{A}$ is an s-(normal) subgroup of $\mathcal{B}$ iff $\mathcal{A}$ is an s-subset of $\mathcal{B}$.
(4) For any s-group $\mathcal{G}$ and for any pair of s-subsets $\mathcal{H}, \mathcal{K}$ of $\mathcal{G}$ such that $\mathcal{H} \subseteq \mathcal{K}$ and $\mathcal{K}$ is an s-subgroup of $\mathcal{G}$ we have $\mathcal{H}$ is an s-normal subgroup of $\mathcal{G}$ implies $\mathcal{H}$ is an s-normal subgroup of $\mathcal{K}$.

In what follows we introduce the notions of s-homomorphism, s-monomorphism, s-epimorphism, s-isomorphism, pure s-isomorphism, s-kernel etc., generalizing some of the corresponding existing notions of soft homomorphism and study some of the standard properties of (inverse) image of s-(normal) subgroups.
(c) An s-map $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an *s-homomorphism* of s-groups, again denoted by $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, iff (1) both $\mathcal{A}, \mathcal{B}$ are s-groups (2) $F: P(U_A) \to P(U_B)$ is any map such that $F_0 = F|U_A: U_A \to U_B$ is a group homomorphism.

Note: Since any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an s-map, from the definition of s-map $F = P(F_0)$, for any subset C of U_A, $F(C) = F_0(C)$ and so we know F if we know F_0 and vice versa.

Consequently, in all our examples we specify only $f: A \to B$ and $F_0: U_A \to U_B$ from which follow the s-map $\mathcal{F} = (f, F)$.
(e) An s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an *s-monomorphism* of s-groups, again denoted by $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, iff both f is one-one and F is one-one.
(f) An s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an *s-epimorphism* of s-groups, again denoted by $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, iff both f is onto and F is onto.
(g) An s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an *s-isomorphism* of s-groups, again denoted by $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, iff (1) $f: A \to B$ is a map (2) $F: P(U_A) \to P(U_B)$ is any s-map such that $F_0: U_A \to U_B$ is an isomorphism (3) $F_0|\overline{A}a: \overline{A}a \to \overline{B}fc$ is an isomorphism for all $c \in f^{-1}fa$ and for all $a \in A$ (4) $\overline{B}fa = F_0\overline{A}a$ for all $a \in A$.
(h) An s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is a *pure s-isomorphism* of s-groups, again denoted by $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, iff (1) $f: A \to B$ is a bijection (2) $F: P(U_A) \to P(U_B)$ is any s-map such that $F_0: U_A \to U_B$ is an isomorphism (3) $F_0|\overline{A}c: \overline{A}c \to \overline{B}fc$ is an isomorphism (4) $\overline{B}fa = F_0\overline{A}a$ for all $a \in A$.
(i) For an s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups, s-*Kernel* of $\mathcal{F}$, denoted by $Ker(\mathcal{F})$, is defined by the whole s-set $Ker(\mathcal{F}) = \mathcal{K}$, where $K = A$, $P(U_K) = P(Ker(F_0))$ or $U_K = Ker(F_0)$ and $\overline{K}: K \to P(U_K)$ is given by $\overline{K}k = Ker(F_0)$ for all $k \in A$.

Note: Unlike in the crisp set up, $Ker(\mathcal{F})$ need *not* be even an s-subset of $\mathcal{A}$ as shown in the following example:
Example 8: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a\}, \{(a, (0)\}, P(\mathbb{Z}_2))$, $\mathcal{B} = (\{b\}, \{(b, (\overline{0}))\}, P(\mathbb{Z}_2))$, $f = \{(a, b)\}$ and $F_0 = \{(0,0), (1,0)\}$.
Then $\mathcal{F}$ is preserving, $Ker\mathcal{F} = \mathcal{K}$ implies $K = A$, $U_K = KerF_0 = \mathbb{Z}_2$ and $\overline{K}a = KerF_0 = \mathbb{Z}_2 \not\subseteq (0) = \overline{A}a$ or $\mathcal{K} \not\subseteq \mathcal{A}$.

Lemma 4.2 *For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups and for any s-subgroup $\mathcal{C}$ of $\mathcal{A}$, the s-image $\mathcal{FC}$ of $\mathcal{C}$ under $\mathcal{F}$ is an s-subgroup of $\mathcal{B}$, whenever $\mathcal{C}$ is constant on each kernel class.*
Proof: Let $\mathcal{FC}$=$\mathcal{D}$. Then D=fC, $U_D = F_0U_C$ and $\overline{D}d = \overline{B}d \cap (\cup_{c \in f^{-1}d \cap C} F_0\overline{C}c)$ for all $d \in D$.
We show that $\mathcal{D}$ is an s-subgroup of $\mathcal{B}$ or (i) $D \subseteq B$ (ii) U_D is a subgroup of U_B and (iii) $\overline{D}d$ is a subgroup of $\overline{B}d$ for all $d \in D$.
(i): $D = fC \subseteq fA = B$
(ii): Since U_C is a subgroup of U_A and $F_0: U_A \to U_B$ is a group homomorphism, $U_D = F_0U_C$ is a subgroup of U_B.
(iii): Let $c \in f^{-1}d \cap C$. Then $\overline{C}c$ is a subgroup of $\overline{A}c$ implies $F_0\overline{C}c$ is a subgroup of $F_0\overline{A}c$, $F_0\overline{A}c$ is a subgroup of U_B implies $F_0\overline{C}c$ is a subgroup of U_B. Since $\mathcal{C}$ is constant on each Kernel class, $\overline{C}c = \overline{C}a$ for all $c \in f^{-1}fa \cap C$ which implies $\cup_{c \in f^{-1}fa \cap C} F_0\overline{C}c = F_0\overline{C}a$ which is a subgroup of U_B. $\overline{B}d$ is also a subgroup of U_B. Since intersection of subgroups is a subgroup, it follows that $\overline{D}d$ is a subgroup of U_B or $\overline{D}d$ is a subgroup of $\overline{B}d$ or $\mathcal{D}$ is an s-subgroup of $\mathcal{B}$.

The following example shows that the above Lemma is *not* true if $\mathcal{C}$ is *not* constant on each kernel class.
Example 9: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{c_1, c_2\}, \{(c_1, \mathbb{Z}), (c_2, \mathbb{Z})\}, P(\mathbb{Z}))$, $\mathcal{B} = (\{b\}, \{(b, \mathbb{Z})\}, P(\mathbb{Z}))$, $\mathcal{C} = (\{c_1, c_2\}, \{(c_1, 2\mathbb{Z}), (c_2, 3\mathbb{Z})\}, P(\mathbb{Z}))$, $f: A \to B$ be given by $f = \{(c_1, b), (c_2, b)\}$ and F_0 be an identity map. Then $\mathcal{C}$ is *not* constant on the kernel class and $\mathcal{F}$ is preserving.
Let $\mathcal{FC} = \mathcal{D}$. Then $D = f\{(c_1, c_2)\} = \{b\}$, $U_D = F_0U_C = F_0\mathbb{Z} = \mathbb{Z}$ and $\overline{D}b = \overline{B}b \cap (\cup_{c \in f^{-1}b \cap C} F_0\overline{C}c) = \mathbb{Z} \cap (2\mathbb{Z} \cup 3\mathbb{Z}) = 2\mathbb{Z} \cup 3\mathbb{Z} \not\leq \mathbb{Z} = \overline{B}b$. Clearly, $\mathcal{D}$ is an s-subset but *not* an s-subgroup.

Corollary 4.3 *For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups, the following are true:*
(a) $\mathcal{FA}$ always an s-subgroup of $\mathcal{B}$, whenever $\mathcal{A}$ is constant on each kernel class.
(b) $\mathcal{FC}$ always an s-subgroup of both $\mathcal{FA}$ and $\mathcal{B}$, whenever both $\mathcal{C}$ and $\mathcal{A}$ are constant on each kernel class
Proof: (a) It follows from the above Lemma with $\mathcal{C} = \mathcal{A}$.
(b) It follows from (a) above and Theorem 4.10 with $\mathcal{C} = \mathcal{C}$ and $\mathcal{D} = \mathcal{A}$.

As in the crisp set up the s-image of an s-normal subgroup is *not* an s-normal subgroup in all of the co-domain s-group, as shown in Example 12 below. However, again as in the crisp set up, in what follows we show that the s-image of an s-normal subgroup is an s-normal subgroup in the image of the domain s-group under some natural conditions:
Lemma 4.4 *For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups and for any s-normal subgroup $\mathcal{C}$ of $\mathcal{A}$ the s-image $\mathcal{FC}$ of $\mathcal{C}$ under $\mathcal{F}$ is an s-normal subgroup of $\mathcal{FA}$, whenever both $\mathcal{A}$ and $\mathcal{C}$ are constants on each kernel class.*
Proof: It follows from Theorem 4.10 with $\mathcal{C} = \mathcal{C}$ and $\mathcal{D} = \mathcal{A}$.

The following example shows that the above Lemma is *not* true if $\mathcal{A}$ is *not* constant on each kernel class but $\mathcal{C}$ is constant on each kernel class.

Example 10: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A}$=$(\{c_1, c_2\}, \{(c_1, 2\mathbb{Z}), (c_2, 3\mathbb{Z})\}, P(\mathbb{Z}))$, $\mathcal{B}$ = $(\{b\}, \{(b, \mathbb{Z})\}, P(\mathbb{Z}))$, $\mathcal{C}$ = $(\{c_1, c_2\}, \{(c_1, 6\mathbb{Z}), (c_2, 6\mathbb{Z})\}, P(\mathbb{Z}))$, $f: A \to B$ be given by $f = \{(c_1, b), (c_2, b)\}$ and F_0 be the identity map.

Then $\mathcal{A}$ is *not* constant on the kernel class $f^{-1}fc_1$ but $\mathcal{C}$ is constant on each kernel class $f^{-1}fa$ and $\mathcal{F}$ is increasing.

Let $\mathcal{FC} = \mathcal{D}$. Then $D = f\{(c_1, c_2)\} = \{b\}$, $U_D = F_0 U_C = \mathbb{Z}$ and $\overline{D}b = \overline{B}b \cap (\cup_{c \in f^{-1}b \cap C} F_0 \overline{C}c) = \mathbb{Z} \cap (6\mathbb{Z} \cup 6\mathbb{Z}) = 6\mathbb{Z}$.

Let $\mathcal{FA} = \mathcal{E}$. Then $E = f\{(c_1, c_2)\} = \{b\}$, $U_E = F_0 U_A = \mathbb{Z}$ and $\overline{E}b = \overline{B}b \cap (\cup_{a \in f^{-1}b \cap A} F_0 \overline{A}a) = \mathbb{Z} \cap (2\mathbb{Z} \cup 3\mathbb{Z}) = 2\mathbb{Z} \cup 3\mathbb{Z}$.

Clearly, $\mathcal{C}$ is an s-normal subgroup of $\mathcal{A}$, $\mathcal{FC} = \mathcal{D}$ is an s-normal subgroup of $\mathcal{B}$, $\mathcal{FA} \neq \mathcal{B}$ and $\mathcal{FA} = \mathcal{E}$ is *not* even an s-subgroup.

The following example shows that the above Lemma is *not* true if $\mathcal{C}$ is *not* constant on each kernel class but $\mathcal{A}$ is constant on each kernel class.

Example 11: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A}$=$(\{c_1, c_2\}, \{(c_1, \mathbb{Z}), (c_2, \mathbb{Z})\}, P(\mathbb{Z}))$, $\mathcal{B}$ = $(\{b\}, \{(b, \mathbb{Z})\}, P(\mathbb{Z}))$, $\mathcal{C}$ = $(\{c_1, c_2\}, \{(c_1, 2\mathbb{Z}), (c_2, 3\mathbb{Z})\}, P(\mathbb{Z}))$, $f: A \to B$ be given by $f = \{(c_1, b), (c_2, b)\}$ and F_0 be the identity map.

Then $\mathcal{C}$ is *not* constant on the kernel class but $\mathcal{A}$ is constant on each kernel class and $\mathcal{F}$ is preserving.

Let $\mathcal{FC} = \mathcal{D}$. Then $D = f\{(c_1, c_2)\} = \{b\}$, $U_D = F_0 U_C = \mathbb{Z}$ and $\overline{D}b = \overline{B}b \cap (\cup_{c \in f^{-1}b \cap C} F_0 \overline{C}c) = \mathbb{Z} \cap (2\mathbb{Z} \cup 3\mathbb{Z}) = 2\mathbb{Z} \cup 3\mathbb{Z}$.

Let $\mathcal{FA} = \mathcal{E}$. Then $E = f\{(c_1, c_2)\} = \{b\}$, $U_E = F_0 U_A = \mathbb{Z}$ and $\overline{E}b = \overline{B}b \cap (\cup_{a \in f^{-1}b \cap A} F_0 \overline{A}a) = \mathbb{Z} \cap (\mathbb{Z} \cup \mathbb{Z}) = \mathbb{Z}$.

Clearly, $\mathcal{C}$ is an s-normal subgroup of $\mathcal{A}$ but $\mathcal{FC} = \mathcal{D}$ is *not* even an s-subgroup.

Note: For any epimorphism of groups $f: G \to H$, fG is trivially a normal subgroup of H.

The following example shows that even for an s-epimorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups and even when $\mathcal{A}$ is constant on each kernel class, $\mathcal{FA}$ need *not* be an s-normal subgroup of $\mathcal{B}$.

Example 12: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A}$ = $(\{a\}, \{(a, H_2)\}, P(H_2))$, $\mathcal{B}$ = $(\{b\}, \{(b, A_4)\}, P(A_4))$, $f: A \to B$ is given by $f = \{(a, b)\}$ and F_0 be the inclusion homomorphism. Then $\mathcal{F}$ is *not* decreasing.

Let $\mathcal{FA} = \mathcal{D}$. Then $D = fA = \{b\}$, $U_D = F_0 U_A = A_4$ and $\overline{D}b = \overline{B}b \cap (\cup_{c \in f^{-1}b \cap C} F_0 \overline{A}c) = A_4 \cap H_2 = H_2$ is *not* a normal subgroup of $\overline{B}b = A_4$.

Therefore $\mathcal{D} = \mathcal{FA}$ is *not* an s-normal subgroup of $\mathcal{B}$.

In what follows we show that for an s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, $\mathcal{FA}$ actually equals $\mathcal{B}$ whenever $\mathcal{F}$ is an s-epimorphism and $\mathcal{F}$ is decreasing:

Lemma 4.5 *For any s-epimorphism, $\mathcal{F}_d: \mathcal{A} \to \mathcal{B}$ of s-groups, $\mathcal{F}_d\mathcal{A} = \mathcal{B}$.*

Proof: Let $\mathcal{FA}$=$\mathcal{D}$. Then D=fA, U_D=$F_0 U_A$ and $\overline{D}d = \overline{B}d \cap (\cup_{c \in f^{-1}d \cap A} F_0 \overline{A}c)$ for all $d \in D$.

We show that $\mathcal{D} = \mathcal{B}$ or (i) $D = B$ (ii) $U_D = U_B$ and (iii) $\overline{D}d = \overline{B}d$ for all $d \in D$

(i): Since f is onto, $D = fA = B$.

(ii): Since F_0 is onto, $U_D = F_0 U_A = U_B$.

(iii): Let $a \in f^{-1}d \cap A$. Then $\overline{D}d = \overline{B}d \cap (\cup_{c \in f^{-1}d \cap A} F_0 \overline{A}c) = \cup_{c \in f^{-1}d \cap A} (\overline{B}d \cap F_0 \overline{A}c)$. Since $\mathcal{F}$ is decreasing, for all $c \in f^{-1}d \cap A$, $\overline{B}d = \overline{B}fc \subseteq F_0 \overline{A}c$ which implies $F_0 \overline{A}c \cap \overline{B}d = \overline{B}d$, which in turn implies $\overline{D}d = \overline{B}d$ or $\mathcal{D} = \mathcal{B}$.

The following example shows that the above Lemma is *not* true if $\mathcal{F}$ is an s-epimorphism, but $\mathcal{F}$ is *not* decreasing.

Example 13: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A}$ = $(\{a\}, \{(a, Z_2)\}, P(\mathbb{Z}_4))$, $\mathcal{B}$=$(\{b\}, \{(b, \mathbb{Z}_4)\}, P(\mathbb{Z}_4))$, $f: A \to B$ be given by f=$\{(a, b)\}$ and F_0=$1_{\mathbb{Z}_4}$, where $Z_2 = \{\overline{0}, \overline{2}\}$.

Then $\mathcal{F}$ is onto, $\overline{B}fa = \overline{B}b = \mathbb{Z}_4 \supseteq Z_2 = F_0 \overline{A}a$, implying $\mathcal{F}$ is *not* decreasing.

Let $\mathcal{FA} = \mathcal{D}$. Then $D = fA = \{b\}$, $U_D = F_0 U_A = F_0 \mathbb{Z}_4 = \mathbb{Z}_4$ and $\overline{D}b = \overline{B}b \cap (\cup_{c \in f^{-1}d \cap A} F_0 \overline{A}c) = \mathbb{Z}_4 \cap Z_2 = Z_2$, $\overline{D}b = Z_2 \neq \mathbb{Z}_4 = \overline{B}b$, implying $\mathcal{FA} \neq \mathcal{B}$.

The following example shows that the above Lemma is *not* true if $\mathcal{F}$ is decreasing but f is onto and F_0 is *not* onto.

Example 14: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a\}, \{(a, \mathbb{Z}_2)\}, P(\mathbb{Z}_2))$, $\mathcal{B} = (\{b\}, \{(b, (0))\}, P(\mathbb{Z}_4))$, $f: A \to B$ be given by $f = \{(a, b)\}$ and F_0 be the inclusion map $F_0 = \{(0,0), (1,2)\}$, where $Z_2 = \{\overline{0}, \overline{2}\}$.

Then F_0 is not onto, $\overline{B}fa = \overline{B}b = (0) \subseteq \mathbb{Z}_2 = F_0\overline{A}a$ or $\mathcal{F}$ is decreasing.

Let $\mathcal{F}\mathcal{A} = \mathcal{D}$. Then $D = fA = \{b\}$, $U_D = F_0 U_A = F_0 \mathbb{Z}_2 = Z_2$ and $\overline{D}b = \overline{B}b \cap (\cup_{c \in f^{-1}d \cap A} F_0\overline{A}c) = (0) \cap Z_2 = (0)$. Since $U_D = Z_2 \neq \mathbb{Z}_4 = U_B$, implying $\mathcal{F}_d\mathcal{A} \neq \mathcal{B}$.

The following example shows that the above Lemma is *not* true if $\mathcal{F}$ is decreasing, but f is *not* onto and F_0 is onto.

Example15: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ an s-homomorphism given by: $\mathcal{A} = (\{a_1, a_2\}, \{(a_1, \mathbb{Z}_2), (a_2, \mathbb{Z}_2)\}, P(\mathbb{Z}_4))$, $\mathcal{B} = (\{b_1, b_2\}, \{(b_1, (0)), (b_2, \mathbb{Z}_2)\}, P(\mathbb{Z}_4))$, $f: A \to B$ be given by $f = \{(a_1, b_1), (a_2, b_1)\}$ and F_0 be the identity map.

Then f is *not* onto, $\overline{B}fa_1 = \overline{B}b_1 = (0) \subseteq \mathbb{Z}_2 = F_0\overline{A}a_1$ and $\overline{B}fa_2 = \overline{B}b_1 = (0) \subseteq \mathbb{Z}_2 = F_0\overline{A}a_2$, implying $\mathcal{F}$ is decreasing.

Let $\mathcal{F}\mathcal{A} = \mathcal{D}$. Then $D = fA = f\{a_1, a_2\} = \{b_1\} \neq \{b_1, b_2\} = B$, implying $\mathcal{D} \neq \mathcal{B}$ or $\mathcal{F}_d\mathcal{A} \neq \mathcal{B}$.

Lemma 4.6 *For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups and for any s-subgroup $\mathcal{D}$ of $\mathcal{B}$, the s-inverse image, $\mathcal{F}^{-1}\mathcal{D}$ of $\mathcal{D}$ under $\mathcal{F}$ is an s-subgroup of both $\mathcal{A}$ and $\mathcal{F}^{-1}B$.*

Proof: Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{C}$. Then $C = f^{-1}D$, $U_C = F_0^{-1}U_D$ and $\overline{C}c = \overline{A}c \cap F_0^{-1}\overline{D}fc$ for all $c \in C$.

We show that $\mathcal{C}$ is an s-subgroup of $\mathcal{A}$ or (i) $C \subseteq A$ (ii) U_C is a subgroup of U_A and (iii) $\overline{C}c$ is a subgroup of $\overline{A}c$ for all $c \in C$.

(i): $C = f^{-1}D \subseteq f^{-1}B = A$

(ii): Since U_D is a subgroup of U_B and F_0 is homomorphism, $U_C = F_0^{-1}U_D$ is a subgroup of $F_0^{-1}U_B = U_A$.

(iii): Let $c \in C = f^{-1}D$. Then $fc \in D$, $\overline{D}fc$ is a subgroup of $\overline{B}fc$ implies $F_0^{-1}\overline{D}fc$ is a subgroup of $F_0^{-1}\overline{B}fc$. $\overline{B}fc$ is a subgroup of U_B implies $F_0^{-1}\overline{B}fc$ is a subgroup of $F_0^{-1}U_B = U_A$. By transitivity, $F_0^{-1}\overline{D}fc$ is a subgroup of U_A which implies $\overline{A}c \cap F_0^{-1}\overline{D}fc$ is a subgroup of U_A or $\overline{C}c$ is a subgroup of $\overline{A}c$ or $\mathcal{C}$ is an s-subgroup of $\mathcal{A}$.

The fact that $\mathcal{F}^{-1}\mathcal{D}$ is an s-subgroup of $\mathcal{F}^{-1}B$ follows from Theorem 4.11 with $\mathcal{C} = \mathcal{D}$ and $\mathcal{D} = \mathcal{B}$.

Note: For any homomorphism of groups $f: A \to B$, $f^{-1}B$ is equal to A. However, in the case of s-homomorphisms $\mathcal{F}^{-1}\mathcal{B}$ can be a proper s-subgroup. But we always have

Corollary 4.7 *For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups, $\mathcal{F}^{-1}\mathcal{B}$ always an s-subgroup of $\mathcal{A}$.*

The following Example shows that in general $\mathcal{F}^{-1}\mathcal{B}$ need *not* equal $\mathcal{A}$ or $\mathcal{F}^{-1}\mathcal{B}$ can be a proper s-subgroup of $\mathcal{A}$:

Example 16: Let $\mathcal{F}_d: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a\}, \{(a, Z_2)\}, P(\mathbb{Z}_4))$, $\mathcal{B} = (\{b\}, \{(b, (\overline{0}))\}, P(\mathbb{Z}_4))$, $f: A \to B$ be given by $f = \{(a, b)\}$, and F_0 be the identity map, where $Z_2 = \{\overline{0}, \overline{2}\}$.

Then $\overline{B}fa = \overline{B}b = (\overline{0}) \subseteq Z_2 = F_0\overline{A}a$, implying $\mathcal{F}$ is *not* increasing.

Let $\mathcal{F}^{-1}\mathcal{B} = \mathcal{C}$. Then $C = f^{-1}B = A$, $U_C = F_0^{-1}U_B = F_0^{-1}\mathbb{Z}_4 = \mathbb{Z}_4$ and $\overline{C}a = \overline{A}a \cap F_0^{-1}\overline{B}fa = Z_2 \cap F_0^{-1}(\overline{0}) = (\overline{0}) \subsetneq Z_2 = \overline{A}a$, implying $\mathcal{F}_d^{-1}\mathcal{B} \subsetneq \mathcal{A}$.

In what follows we show that whenever $\mathcal{F}$ is an increasing s-homomorphism $\mathcal{F}_i: \mathcal{A} \to \mathcal{B}$, $\mathcal{F}_i^{-1}\mathcal{B}$ equals $\mathcal{A}$.

Lemma 4.8 For any s-homomorphism $\mathcal{F}_i: \mathcal{A} \to \mathcal{B}$ of s-groups, $\mathcal{F}_i^{-1}\mathcal{B} = \mathcal{A}$.

Proof: Let $\mathcal{F}^{-1}\mathcal{B} = \mathcal{C}$. Then $C = f^{-1}B$, $U_C = F_0^{-1}U_B$ and $\overline{C}c = \overline{A}c \cap F_0^{-1}\overline{B}fc$ for all $c \in C$.

We show that $\mathcal{C} = \mathcal{A}$ or (i) $C = A$ (ii) $U_C = U_A$ and (iii) $\overline{C}c = \overline{A}c$ for all $c \in C$.

(i): $C = f^{-1}B = A$.

(ii): $U_C = F_0^{-1}U_B = U_A$.

(iii): Let $c \in C$ be fixed. Since $\mathcal{F}$ is increasing, $F_0\overline{A}c \subseteq \overline{B}fc$, since $\phi M \subseteq N$ implies $M \subseteq \phi^{-1}N$, $F_0\overline{A}c \subseteq \overline{B}fc$ implies $\overline{A}c \subseteq F_0^{-1}\overline{B}fc$ or $\overline{A}c \cap F_0^{-1}\overline{B}fc = \overline{A}c$ or $\overline{C}c = \overline{A}c \cap F_0^{-1}\overline{B}fc = \overline{A}c$ or $\mathcal{C} = \mathcal{A}$.

The above Lemma is not true if $\mathcal{F}$ is not increasing, as shown in Example 16 above.

Lemma 4.9 For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups and for any s-normal subgroup $\mathcal{D}$ of $\mathcal{B}$, the s-inverse image $\mathcal{F}^{-1}\mathcal{D}$ of $\mathcal{D}$ under $\mathcal{F}$ is an s-normal subgroup of $\mathcal{F}^{-1}\mathcal{B}$.

Proof: It follows from Theorem 4.11 with $\mathcal{C} = \mathcal{D}$ and $\mathcal{D} = \mathcal{B}$.

Note: In the crisp set up for any group homomorphism $f: G \to H$ always $f^{-1}H$, being equal to G, is trivially a (normal) subgroup of G. However, the following Example shows that the same thing is not true in case of s-homomorphisms of s-groups:

In what follows we construct an s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups in such a way that $\mathcal{F}^{-1}\mathcal{B}$ need not be an s-normal subgroup of $\mathcal{A}$.

Example 17: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a\}, \{(a, A_4)\}, P(A_4))$, $\mathcal{B} = (\{b\}, \{(b, H_2)\}, P(A_4))$, $f: A \to B$ is given by $f = \{(a, b)\}$ and $F_0 = 1_{A_4}$. Then $\mathcal{F}$ is decreasing.

Let $\mathcal{F}^{-1}\mathcal{B} = \mathcal{C}$. Then $C = f^{-1}B = A$, $U_C = F_0^{-1}U_B = A_4$ and $\overline{C}a = \overline{A}a \cap F_0^{-1}\overline{B}fa = A_4 \cap F_0^{-1}\overline{B}b = A_4 \cap H_2 = H_2$. $\overline{C}a = H_2$ is not a normal subgroup of $A_4 = \overline{A}a$ or $\mathcal{F}^{-1}\mathcal{B} = \mathcal{C}$ is not an s-normal subgroup of $\mathcal{A}$.

Theorem 4.10 For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups and for any pair of s-subsets $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{A}$ such that $\mathcal{C}$ is an s-(normal) subgroup of $\mathcal{D}$, we have $\mathcal{F}\mathcal{C}$ is an s-(normal) subgroup of $\mathcal{F}\mathcal{D}$, whenever both $\mathcal{C}$ and $\mathcal{D}$ are constants on each kernel class.

Proof: Let $\mathcal{F}\mathcal{C} = \mathcal{M}$. Then $M = fC$, $U_M = F_0U_C$ and $\overline{M}m = \overline{B}m \cap (\cup_{c \in f^{-1}m \cap C} F_0\overline{C}c)$ for all $m \in M$.

Let $\mathcal{F}\mathcal{D} = \mathcal{N}$. Then $N = fD$, $U_N = F_0U_D$ and $\overline{N}n = \overline{B}n \cap (\cup_{d \in f^{-1}n \cap D} F_0\overline{D}d)$ for all $n \in N$.

$\mathcal{C} \subseteq \mathcal{D}$ implies $C \subseteq D$, $U_C \subseteq U_D$ and $\overline{C}c \subseteq \overline{D}c$ for all $c \in C$.

We show that $\mathcal{M}$ is an s-(normal) subgroup of $\mathcal{N}$ or (i) $M \subseteq N$ (ii) U_M is a (normal) subgroup of U_N and (iii) $\overline{M}m$ is a (normal) subgroup of $\overline{N}m$ for all $m \in M$.

(i): $C \subseteq D$ implies $M = fC \subseteq fD = N$.

(ii): U_C is a (normal) subgroup of U_D implies $U_M = F_0U_C$ is a (normal) subgroup of $F_0U_D = U_N$, by 2(B)(ii).

(iii): Let $m \in M \subseteq N$. Then $m = fc$ for some $c \in C \subseteq D$. Since $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{F}$-constants $\overline{C}a = \overline{C}c$ for all $a \in f^{-1}fc \cap C$ and $\overline{D}b = \overline{D}c$ for all $b \in f^{-1}fc \cap D$.

$C \subseteq D$ implies $f^{-1}fc \cap C \subseteq f^{-1}fc \cap D$, $\cup_{a \in f^{-1}fc \cap C} F_0\overline{C}a = F_0\overline{C}c$ and $\cup_{b \in f^{-1}fc \cap D} F_0\overline{D}b = F_0\overline{D}c$. $\overline{C}c$ is a (normal) subgroup of $\overline{D}c$ for all $c \in C$ implies $F_0\overline{C}c$ is a (normal) subgroup of $F_0\overline{D}c$ which in turn implies, by 2(B)(ii), $\overline{M}m = \overline{M}fc = \overline{B}fc \cap (\cup_{c \in f^{-1}fc \cap C} F_0\overline{C}c)$ is a (normal) subgroup of $\overline{B}fc \cap (\cup_{d \in f^{-1}fc \cap D} F_0\overline{D}d) = \overline{N}fc$ or $\mathcal{F}\mathcal{C}$ is an s-(normal) subgroup of $\mathcal{F}\mathcal{D}$.

The following example shows that the above Theorem is not true if $\mathcal{C}$ is not constant on each kernel class but $\mathcal{D}$ is constant on each kernel class.

Example 18: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a_1, a_2\}, \{(a_1, \mathbb{Z}), (a_2, \mathbb{Z})\}, P(\mathbb{Z}))$, $\mathcal{B} = (\{b\}, \{(b, \mathbb{Z})\}, P(\mathbb{Z})) = \mathcal{D}$, $\mathcal{C} = (\{a_1, a_2\}, \{(a_1, 2\mathbb{Z}), (a_2, 3\mathbb{Z})\}, P(\mathbb{Z}))$, $f: A \to B$ be given by $f = \{(a_1, b), (a_2, b)\}$ and F_0 be the identity map.

Then $\mathcal{C}$ is not constant on the kernel class $f^{-1}fa_1$ but $\mathcal{D}$ is constant on each kernel class $f^{-1}fa$.

Let $\mathcal{F}\mathcal{C} = \mathcal{M}$. Then $M = f\{(a_1, a_2)\} = \{b\}$, $U_M = F_0U_C = \mathbb{Z}$ and $\overline{M}b = \overline{B}b \cap (\cup_{c \in f^{-1}b \cap C} F_0\overline{C}c) = \mathbb{Z} \cap (2\mathbb{Z} \cup 3\mathbb{Z}) = 2\mathbb{Z} \cup 3\mathbb{Z}$.

Let $\mathcal{F}\mathcal{D} = \mathcal{N}$. Then $N = f\{(a_1, a_2)\} = \{b\}$, $U_N = F_0U_D = \mathbb{Z}$ and $\overline{N}b = \overline{B}b \cap (\cup_{a \in f^{-1}b \cap A} F_0\overline{D}b) = \mathbb{Z} \cap \mathbb{Z} = \mathbb{Z}$.

Clearly, $\mathcal{C}$, $\mathcal{D}$ are an s-(normal) subgroups of $\mathcal{A}$, $\mathcal{F}\mathcal{D} = \mathcal{N}$ is an s-normal subgroup of $\mathcal{A}$, $\mathcal{F}\mathcal{C} = \mathcal{M}$ is not even an s-subgroup.

The following example shows that the above Theorem is not true if $\mathcal{C}$ is constant on each kernel class but $\mathcal{D}$ is not constant on each kernel class.

Example 19: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a\}, \{(a, \mathbb{Z})\}, P(\mathbb{Z})) = \mathcal{C}$, $\mathcal{B} = (\{b_1, b_2\}, \{(b_1, \mathbb{Z})\}, P(\mathbb{Z}))$, $\mathcal{D} = (\{b_1, b_2\}, \{(b_1, 2\mathbb{Z}), (b_2, 3\mathbb{Z})\}, P(\mathbb{Z}))$, $f: A \to B$ be given by $f = \{(a, b_1), (a, b_2)\}$ and F_0 be the identity map.

Then $\mathcal{C}$ is constant on the kernel class but $\mathcal{D}$ is not constant on each kernel class.

Let $\mathcal{F}\mathcal{C} = \mathcal{M}$. Then $M = f\{(a)\} = \{b_1, b_2\}$, $U_M = F_0U_C = \mathbb{Z}$ and $\overline{M}b_1 = \overline{B}b_1 \cap (\cup_{c \in f^{-1}b \cap C} F_0\overline{C}c) = \mathbb{Z} \cap \mathbb{Z} = \mathbb{Z} = \overline{M}b_2$.

Let $\mathcal{F}\mathcal{D} = \mathcal{N}$. Then $N = f\{(b_1, b_2)\} = \{a\}$, $U_N = F_0U_D = \mathbb{Z}$ and $\overline{N}a = \overline{B}a \cap (\cup_{a \in f^{-1}b \cap A} F_0\overline{D}a) = \mathbb{Z} \cap (2\mathbb{Z} \cup 3\mathbb{Z}) = 2\mathbb{Z} \cup 3\mathbb{Z}$.

Clearly, $\mathcal{C}$, $\mathcal{D}$ are an s-normal subgroup of $\mathcal{A}$ but $\mathcal{F}\mathcal{D} = \mathcal{N}$ is not even an s-subgroup.

Theorem 4.11 For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups and for any pair of s-subsets $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{B}$ such that $\mathcal{C}$ is an s-(normal) subgroup of $\mathcal{D}$, we have $\mathcal{F}^{-1}C$ is an s-(normal) subgroup of $\mathcal{F}^{-1}D$.

Proof: Let $\mathcal{F}^{-1}\mathcal{C} = \mathcal{M}$. Then $M = f^{-1}C$, $U_M = F_0^{-1}U_C$ and $\overline{M}m = \overline{A}m \cap F_0^{-1}\overline{C}fm$ for all $m \in M$.

Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{N}$. Then $N = f^{-1}D$, $U_N = F_0^{-1}U_D$ and $\overline{N}n = \overline{A}n \cap F_0^{-1}\overline{D}fn$ for all $n \in N$.

$\mathcal{C} \subseteq \mathcal{D}$ implies $C \subseteq D$, $U_C \subseteq U_D$ and $\overline{C}c \subseteq \overline{D}c$ for all $c \in C$.

We show that $\mathcal{M}$ is an s-(normal) subgroup of $\mathcal{N}$ or (i) $M \subseteq N$ (ii) U_M is a (normal) subgroup of U_N and (iii) $\overline{M}m$ is a (normal) subgroup of $\overline{N}m$ for all $m \in M$.
(i): $C \subseteq D$ implies $M = f^{-1}C \subseteq f^{-1}D = N$.
(ii): U_C is a (normal) subgroup of U_D implies by 2(B)(ii), $U_M = F_0^{-1}U_C$ is a (normal) subgroup of $F_0^{-1}U_D = U_N$.
(iii): Let $m \in M$. Then $fm \in C$, $\overline{C}c$ is a (normal) subgroup of $\overline{D}c$ for all $c \in C$, by 2(B)(ii) implies $F_0^{-1}\overline{C}c$ is a (normal) subgroup of $F_0^{-1}\overline{D}c$ which in turn implies again by 2(B)(ii), $\overline{M}m = \overline{A}m \cap F_0^{-1}\overline{C}fm$ is a (normal) subgroup of $\overline{A}m \cap F_0^{-1}\overline{D}fm = \overline{N}m$ or $\mathcal{F}^{-1}C$ is an s-(normal) subgroup of $\mathcal{F}^{-1}D$.

Lemma 4.12 For any s-homomorphism $\mathcal{F}_i: \mathcal{A} \to \mathcal{B}$ of s-groups and for any s-(normal) subgroup $\mathcal{C}$ of $\mathcal{A}$ which is constant on each kernel class, $\mathcal{C}$ is an s-(normal) subgroup of $\mathcal{F}_i^{-1}\mathcal{F}_i\mathcal{C}$.
Proof: Let $\mathcal{FC} = \mathcal{D}$. Then $D=fC$, $U_D=F_0U_C$ and $\overline{D}d=\overline{B}d \cap (\cup_{c\in f^{-1}d\cap C} F_0\overline{C}c)$ for all $d \in D$.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{E}$. Then $E = f^{-1}D$, $U_E = F_0^{-1}U_D$ and $\overline{E}e = \overline{A}e \cap F_0^{-1}\overline{D}fe$ for all $e \in E$.
(a): We show that $\mathcal{C}$ is an s-(normal) subgroup of $\mathcal{E}$ or (i) $C \subseteq E$ (ii) U_C is a (normal) subgroup of U_E and (iii) $\overline{C}c$ is a (normal) subgroup of $\overline{E}c$ for all $c \in C$.
(i): $C \subseteq f^{-1}(fC) = f^{-1}D = E$.
(ii): $U_C \subseteq F_0^{-1}(F_0U_C) = F_0^{-1}U_D = U_E$. Clearly, from 2(B)(i) U_C is a (normal) subgroup of U_E.
(iii): Let $c \in C$. Then $\overline{E}c=\overline{A}c \cap F_0^{-1}\overline{D}fc = \overline{A}c \cap F_0^{-1}(\overline{B}fc \cap (\cup_{a\in f^{-1}fc\cap C} F_0\overline{C}a))$. Since $\mathcal{C}$ is constant on each kernel class, $\overline{C}a = \overline{C}c$ for all $a \in f^{-1}fc \cap C$ or $\cup_{a\in f^{-1}fc\cap C} F_0\overline{C}a = F_0\overline{C}c$ or $\overline{E}c = \overline{A}c \cap F_0^{-1}(\overline{B}fc \cap F_0\overline{C}c)$.
Since $\mathcal{F}$ is increasing and $\mathcal{C}$ is an s-subset of $\mathcal{A}$, $F_0\overline{C}c \subseteq F_0\overline{A}c \subseteq \overline{B}fc$ which implies $F_0\overline{C}c \subseteq \overline{B}fc$ or $F_0\overline{C}c \cap \overline{B}fc = F_0\overline{C}c$, $\overline{E}c = \overline{A}c \cap F_0^{-1}(F_0\overline{C}c)$.
Again since $\overline{C}c \subseteq F_0^{-1}F_0\overline{C}c$, $\overline{C}c = \overline{C}c \cap \overline{A}c \subseteq F_0^{-1}F_0\overline{C}c \cap \overline{A}c = \overline{E}c$ or $\overline{C}c$ is a subset of $\overline{E}c$.
From 4.2, $\mathcal{D}$ is an s-subgroup which implies $\overline{D}fc$ is a subgroup which with $\mathcal{A}$ is an s-group, inverse image of subgroup is a subgroup imply $\overline{A}c \cap F_0^{-1}\overline{D}fc = \overline{E}c$ is a subgroup. Now 2(B)(i), $\overline{C}c$ is a normal subgroup of $\overline{A}c$, $\overline{C}c \subseteq \overline{E}c$ imply $\overline{C}c$ is a (normal) subgroup of $\overline{E}c$.

The following example shows that the above Lemma is not true if $\mathcal{F}$ is not increasing but $\mathcal{C}$ is constant on each kernel class.
Example 20: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism defined by: $\mathcal{A} = (\{a\}, \{(a, \mathbb{Z}_4)\}, P(\mathbb{Z}_4)) = \mathcal{C}$, $\mathcal{B} = (\{b\}, \{(b, (0))\}, P(\mathbb{Z}_4))$, $f = \{(a, b)\}$ and $F_0 = 1_{\mathbb{Z}_4}$.
Then $\mathcal{C}$ is an s-(normal) subgroup of $\mathcal{A}$, $\overline{B}fa = \overline{B}b = (0) \subseteq \mathbb{Z}_4 = F_0\overline{A}a$, implying that $\mathcal{F}$ is not increasing.
Further, $\mathcal{C}$ is constant on each kernel class $f^{-1}fa = f^{-1}\{b\} = \{a\}$, $\overline{C}a = \mathbb{Z}_4$.
Let $\mathcal{FC} = \mathcal{D}$. Then $D = fC = \{b\}$, $U_D = F_0U_C = F_0\mathbb{Z}_4 = \mathbb{Z}_4$ and $\overline{D}b = \overline{B}b \cap (\cup_{c\in f^{-1}b\cap C} F_0\overline{C}c) = (0) \cap \mathbb{Z}_4 = (\bar{0})$
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{E}$. Then $E = f^{-1}D = \{a\} = C$, $U_E = F_0^{-1}U_D = \mathbb{Z}_4 = U_C$ and $\overline{E}a = \overline{A}a \cap F_0^{-1}\overline{D}fa = \mathbb{Z}_4 \cap (0) = (\bar{0}) \subsetneq \mathbb{Z}_4 = \overline{C}a$, implying $\mathcal{C} \not\subseteq \mathcal{E}$.

Notice that in the above example $\mathcal{F}$ is an s-monomorphism, however $\mathcal{F}^{-1}\mathcal{FC} \neq \mathcal{C}$ (cf. 4.14) as $\mathcal{F}$ is not increasing.

The following example shows that the above Lemma is not true if $\mathcal{F}$ is increasing but $\mathcal{C}$ is not constant on a kernel class.
Example 21: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism defined by: $\mathcal{A}=(\{a_1, a_2\}, \{(a_1, Z_2), (a_2, \mathbb{Z}_4)\}, P(\mathbb{Z}_4))$, $\mathcal{B}=(\{b\}, \{(b, \mathbb{Z}_4)\}, P(\mathbb{Z}_4))$, $\mathcal{C}=(\{a_1, a_2\}, \{(a_1, (0)), (a_2, Z_2)\}, P(\mathbb{Z}_4))$, $f = \{(a_1, b), (a_2, b)\}$ *and* F_0 *be the identity map on* $\mathbb{Z}_4$. *Here* $Z_2 = \{\bar{0}, \bar{2}\}$.
Then $\mathcal{C}$ is an s-(normal) subgroup of $\mathcal{A}$, $\overline{B}fa_1 = \overline{B}b = \mathbb{Z}_4 \supseteq Z_2 = F_0\overline{A}a_1$, $\overline{B}fa_2 = \overline{B}b = \mathbb{Z}_4 = F_0\overline{A}a_2$, so that $\mathcal{F}$ is increasing.
Further, $\mathcal{C}$ is not constant on the kernel class $f^{-1}fa_1 = f^{-1}b = \{a_1, a_2\}$, as $(0) = \overline{C}a_1$ and $Z_2 = \overline{C}a_2$.
Let $\mathcal{FC} = \mathcal{D}$. Then $D = fc = f\{(a_1, a_2)\} = \{b\}$, $U_D = F_0U_C = F_0\mathbb{Z}_4 = \mathbb{Z}_4$ and $\overline{D}b = \overline{B}b \cap (\cup_{c\in f^{-1}b\cap C} F_0\overline{C}c) = \mathbb{Z}_4 \cap (F_0\overline{C}a_1 \cup F_0\overline{C}a_2) = \mathbb{Z}_4 \cap (0) \cup Z_2) = Z_2$.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{E}$. Then $E = f^{-1}D = f^{-1}b = \{(a_1, a_2)\}$, $U_E = F_0^{-1}U_D = \mathbb{Z}_4$ and $\overline{E}a_1 = \overline{A}a_1 \cap F_0^{-1}\overline{D}fa_1 = Z_2 \cap Z_2 = Z_2 \not\subseteq (0) = \overline{C}a_1$, implying $\mathcal{E} \not\subseteq \mathcal{C}$.

Lemma 4.13 For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups and for any s-(normal) subgroups $\mathcal{C}$ of $\mathcal{A}$, $\mathcal{F}^{-1}\mathcal{FC}$ is an s-(normal) subgroup of $\mathcal{F}^{-1}\mathcal{FA}$, whenever both $\mathcal{C}$ and $\mathcal{A}$ are constants on each kernel class.

Proof: Since $\mathcal{C}$, $\mathcal{A}$ are constants on each kernel class and $\mathcal{C}$ is an s-(normal) subgroup of $\mathcal{A}$, by 4.10 we have $\mathcal{FC}$ is an s-(normal) subgroup of $\mathcal{FA}$ and again by 4.11 we have $\mathcal{F}^{-1}\mathcal{FC}$ is an s-(normal) subgroup of $\mathcal{F}^{-1}\mathcal{FA}$.

Lemma 4.14 For any s-homomorphism $\mathcal{F}_i: \mathcal{A} \to \mathcal{B}$ of s-groups, and for any s-(normal) subgroup $\mathcal{C}$ of $\mathcal{A}$ such that Ker$\mathcal{F} \subseteq \mathcal{C}$, $\mathcal{F}_i^{-1}\mathcal{F}_i\mathcal{C} = \mathcal{C}$, whenever $\mathcal{C}$ is constant on each kernel class.
Proof: Let $\mathcal{FC}$=$\mathcal{D}$. Then D=fC, U_D=F_0U_C and $\overline{D}d = \overline{B}d \cap (\cup_{c\in f^{-1}d\cap C} F_0\overline{C}c)$ for all d ∈ D.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{E}$. Then E = f^{-1}D, $U_E = F_0^{-1}U_D$ and $\overline{E}e = \overline{A}e \cap F_0^{-1}\overline{D}fc$ for all e ∈ E.
Ker$\mathcal{F} \subseteq \mathcal{C}$ implies K = C = A, U_K = Ker$F_0 \subseteq U_C$ and $\overline{K}k$ = Ker$F_0 \subseteq \overline{C}k$ for all k ∈ A.
We show that $\mathcal{C} = \mathcal{E}$ or (i) E = C (ii) $U_E = U_C$ and (iii) $\overline{E}e = \overline{C}e$ for all e ∈ E
(i): E = f^{-1}D = f^{-1}fC = f^{-1}(fA) = A = C.
(ii): $U_E = F_0^{-1}U_D = F_0^{-1}(F_0U_C) = U_C$, where the last equality is due to the fact that Ker$F_0 \subseteq U_C$ and by 2(B)(ii).
(iii): Let e ∈ E = C be fixed. Then $\overline{E}e = \overline{A}e \cap F_0^{-1}\overline{D}fe = \overline{A}e \cap F_0^{-1}(\overline{B}fe \cap (\cup_{c\in f^{-1}fe\cap C} F_0\overline{C}c)) = \overline{A}e \cap F_0^{-1}\overline{B}fe \cap F_0^{-1}(\cup_{c\in f^{-1}fe\cap C} F_0\overline{C}c)$.
Since $\mathcal{C}$ is constant on each kernel class, $\overline{C}c = \overline{C}e$ for all c ∈ f^{-1}fe ∩ C implies $\cup_{c\in f^{-1}fe\cap C} F_0\overline{C}c = F_0\overline{C}e$ which implies $\overline{E}e = \overline{A}e \cap F_0^{-1}\overline{B}fe \cap F_0^{-1}(F_0\overline{C}e) \overset{(2)}{=} \overline{A}e \cap F_0^{-1}\overline{B}fe \cap \overline{C}e \overset{(3)}{=} F_0^{-1}\overline{B}fe \cap \overline{C}e$, where the second equality is due to the fact that Ker$F_0 \subseteq \overline{C}e$ and the third equality is due to $\mathcal{C} \subseteq \mathcal{A}$.
Now since $\mathcal{F}$ is increasing and $\mathcal{C}$ is an s-subgroup of $\mathcal{A}$, $F_0\overline{C}e \subseteq F_0\overline{A}e \subseteq \overline{B}fe$ or $F_0\overline{C}e \subseteq \overline{B}fe$ which implies $\overline{C}e \subseteq F_0^{-1}\overline{B}fe$ or $\overline{C}e \cap F_0^{-1}\overline{B}fe = \overline{C}e$. Therefore $\overline{E}e = \overline{C}e \cap F_0^{-1}\overline{B}fe = \overline{C}e$ or $\mathcal{C} = \mathcal{E}$.

The following example shows that the above Lemma is not true if $\mathcal{F}$ is not increasing but Ker$\mathcal{F} \subseteq \mathcal{C}$ and $\mathcal{C}$ is constant on each kernel class.
Example 22: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism defined by: $\mathcal{A} = (\{a\}, \{(a, \mathbb{Z}_4)\}, P(\mathbb{Z}_4)) = \mathcal{C}$, $\mathcal{B} = (\{b\}, \{(b, (0))\}, P(\mathbb{Z}_4))$, f = {(a, b)} and $F_0 = 1_{\mathbb{Z}_4}$.
Then $\mathcal{C}$ is an s-(normal) subgroup of $\mathcal{A}$, Ker$\mathcal{F}$=$\mathcal{K}$ =$(\{a\}, \{(a, (0))\}, P((0))) \subseteq \mathcal{C}$ and $\overline{B}fa = \overline{B}b = (0) \subseteq \mathbb{Z}_4 = F_0\overline{A}a$, implying that $\mathcal{F}$ is not increasing.
Further, $\mathcal{C}$ is constant on each Kernel class $f^{-1}fa = f^{-1}\{b\} = \{a\}$, $\overline{C}a = \mathbb{Z}_4$.
Let $\mathcal{FC} = \mathcal{D}$. Then D = fC = {b}, $U_D = F_0U_C = F_0\mathbb{Z}_4 = \mathbb{Z}_4$ and $\overline{D}b = \overline{B}b \cap (\cup_{c\in f^{-1}b\cap C} F_0\overline{C}c) = (0) \cap \mathbb{Z}_4 = (0)$
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{E}$. Then E = f^{-1}D = {a} = C, $U_E = F_0^{-1}U_D = \mathbb{Z}_4 = U_C$ and $\overline{E}a = \overline{A}a \cap F_0^{-1}\overline{D}fa = \mathbb{Z}_4 \cap (0) = (0) \neq \mathbb{Z}_4 = \overline{C}a$, implying $\mathcal{E} \neq \mathcal{C}$ or $\mathcal{F}^{-1}\mathcal{FC} \neq \mathcal{C}$.

The following example shows that the above Lemma is not true if $\mathcal{F}$ is increasing, Ker$\mathcal{F} \subseteq \mathcal{C}$ and $\mathcal{C}$ is not constant on a kernel class.
Example 23: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism defined by:$\mathcal{A}$=$(\{a_1, a_2\}, \{(a_1, Z_2), (a_2, \mathbb{Z}_4)\}, P(\mathbb{Z}_4))$, $\mathcal{B}$=$(\{b\}, \{(b, \mathbb{Z}_4)\}, P(\mathbb{Z}_4))$, $\mathcal{C}$=$(\{a_1, a_2\}, \{(a_1, (0)), (a_2, Z_2)\}, P(\mathbb{Z}_4))$, f = $\{(a_1, b), (a_2, b)\}$ and F_0 be the identity map on $\mathbb{Z}_4$. Here $Z_2 = \{\overline{0}, \overline{2}\}$.
Then $\mathcal{C}$ is an s-(normal) subgroup of $\mathcal{A}$, Ker$\mathcal{F}$=$\mathcal{K}$ = $(\{a_1, a_2\}, \{(a_1, (0)), (a_2, (0))\}, P((0)))$ or Ker$\mathcal{F} \subseteq \mathcal{C}$.
$\overline{B}fa_1 = \overline{B}b = \mathbb{Z}_4 \supseteq Z_2 = F_0\overline{A}a_1$, $\overline{B}fa_2 = \overline{B}b = \mathbb{Z}_4 = F_0\overline{A}a_2$, so that $\mathcal{F}$ is increasing.
Further, $\mathcal{C}$ is not constant on the kernel class $f^{-1}fa_1 = f^{-1}b = \{a_1, a_2\}$, as $(0) = \overline{C}a_1$ and $Z_2 = \overline{C}a_2$.
Let $\mathcal{FC} = \mathcal{D}$. Then D = fC = $f\{(a_1, a_2)\}$ = {b}, $U_D = F_0U_C = F_0\mathbb{Z}_4 = \mathbb{Z}_4$ and $\overline{D}b = \overline{B}b \cap (\cup_{c\in f^{-1}b\cap C} F_0\overline{C}c) = \mathbb{Z}_4 \cap (F_0\overline{C}a_1 \cup F_0\overline{C}a_2) = \mathbb{Z}_4 \cap ((0) \cup Z_2) = Z_2$.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{E}$. Then E = f^{-1}D = f^{-1}b = $\{(a_1, a_2)\}$, $U_E = F_0^{-1}U_D = \mathbb{Z}_4$ and $\overline{E}a_1 = \overline{A}a_1 \cap F_0^{-1}\overline{D}fa_1 = Z_2 \cap Z_2 = Z_2 \neq (0) = \overline{C}a_1$, implying $\mathcal{F}^{-1}\mathcal{FC} = \mathcal{E} \neq \mathcal{C}$.

The following example shows that the above Lemma is not true if $\mathcal{F}$ is increasing, Ker$\mathcal{F} \not\subseteq \mathcal{C}$ but $\mathcal{C}$ is constant on each kernel class.
Example 24: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism defined by: $\mathcal{A}$=$(\{a\}, \{(a, Z_2)\}, P(\mathbb{Z}_4))$, $\mathcal{B}$=$(\{b\}, \{(b, Z_2)\}, P(Z_2))$, $\mathcal{C}$=$(\{a\}, \{(a, (\overline{0}))\}, P(\mathbb{Z}_4))$, f = {(a, b)} and F_0 = {(0,0), (2,0), (1,1), (3,1)}.
Then $\mathcal{C}$ is an s-(normal) subgroup of $\mathcal{A}$, $\overline{B}fa = \overline{B}b = Z_2 \supseteq (0) = F_0\overline{A}a$ or $\mathcal{F}$ is increasing, $\mathcal{K} = (\{a\}, \{(a, Z_2)\}, P(Z_2))$, Ker$(\mathcal{F}) \not\subseteq \mathcal{C}$ and $\mathcal{C}$ is constant on each kernel class $f^{-1}fa = f^{-1}b = \{a\}$, $\overline{C}a = (0)$.
Let $\mathcal{FC} = \mathcal{D}$. Then D = fC = {b}, $U_D = F_0U_C = F_0\mathbb{Z}_4 = Z_2$ and $\overline{D}b = \overline{B}b \cap (\cup_{c\in f^{-1}b\cap C} F_0\overline{C}c) = Z_2 \cap (0) = (\overline{0})$.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{E}$. Then E = f^{-1}D = {a}, $U_E = F_0^{-1}U_D = \mathbb{Z}_4$ and $\overline{E}a = \overline{A}a \cap F_0^{-1}\overline{D}fa = Z_2 \cap Z_2 = Z_2 \neq (0) = \overline{C}a$, implying $\mathcal{F}^{-1}\mathcal{FC} = \mathcal{E} \neq \mathcal{C}$.

Lemma 4.15 For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups such that $\mathcal{A}$ is constant on each kernel class and for any s-subgroup $\mathcal{D}$ of $\mathcal{B}$, $\mathcal{F}\mathcal{F}^{-1}\mathcal{D}$ is an s-subgroup of $\mathcal{D}$.
Proof: Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{C}$. Then $C = f^{-1}D$, $U_C = F_0^{-1}U_D$ and $\overline{C}c = \overline{A}c \cap F_0^{-1}\overline{D}fc$ for all $c \in C$.
Let $\mathcal{F}\mathcal{C} = \mathcal{E}$. Then $E = fC$, $U_E = F_0U_C$ and $\overline{E}e = \overline{B}e \cap (\cup_{c\in f^{-1}e\cap C} F_0\overline{C}c)$ for all $e \in E$.
First observe that $\mathcal{C}$ is constant on each kernel class, by 3.1 as $\mathcal{A}$ is constant on each kernel class. Since $\mathcal{C}$ is constant on each kernel class, clearly $\overline{E}fc = \overline{B}fc \cap F_0\overline{C}c$ for all $c \in C$.
Since s-inverse image of an s-subgroup is an s-subgroup and s-image of an s-subgroup that is constant on each kernel class is an s-subgroup, $\mathcal{C}$ and $\mathcal{E}$ are s-subgroups.
Since for any pair of s-subgroups $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{G}$, $\mathcal{A}$ is an s-subgroup of $\mathcal{B}$ iff $\mathcal{A}$ is an s-subset of $\mathcal{B}$, it is enough to show that $\mathcal{E}$ is an s-subset of $\mathcal{D}$ or (i) $E \subseteq D$ (ii) U_E is a subset of U_D and (iii) $\overline{E}e$ is a subset of $\overline{D}e$ for all $e \in E$.
(i): $E = fC = f(f^{-1}D) \subseteq D$.
(ii): $U_E = F_0U_C = F_0(F_0^{-1}U_D) \subseteq U_D$.
(iii): Let $e \in E \subseteq D$ be fixed. Then $e=fc$ for some $c \in C$, $\overline{E}e=\overline{E}fc \overset{(1)}{=} \overline{B}fc \cap F_0\overline{C}c \overset{(2)}{=} \overline{B}fc \cap F_0(\overline{A}c \cap F_0^{-1}\overline{D}fc) \overset{(3)}{\subseteq} \overline{B}fc \cap F_0\overline{A}c \cap F_0(F_0^{-1}\overline{D}fc) \overset{(4)}{\subseteq} \overline{B}fc \cap F_0(F_0^{-1}\overline{D}fc) \overset{(5)}{\subseteq} \overline{B}fc \cap \overline{D}fc \overset{(6)}{\subseteq} \overline{D}fc \overset{(7)}{=} \overline{D}e$, where (2) is due to the definition of $\overline{C}c$, (3) is due to the fact that $\phi(M \cap N) \subseteq \phi(M) \cap \phi(N)$, and (5) is due to the fact that $\phi\phi^{-1}N \subseteq N$ or $\overline{E}e$ is a subset of $\overline{D}e$ or $\mathcal{E}$ is an s-subset of $\mathcal{D}$.

The following example shows that the containment of $\mathcal{F}\mathcal{F}^{-1}\mathcal{D}$ in $\mathcal{D}$ can be proper in the above Lemma.
Example 25: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a\}, \{(a, Z_2)\}, P(\mathbb{Z}_4))$, $\mathcal{B} = (\{b\}, \{(b, \mathbb{Z}_4)\}, P(\mathbb{Z}_4)) = \mathcal{D}$, $f = \{(a, b)\}$ and F_0 be the identity map, where $Z_2 = \{\overline{0}, \overline{2}\}$.
Then $\mathcal{D}$ is an s-subgroup of $\mathcal{B}$, $\overline{B}fa = \overline{B}b = \mathbb{Z}_4 \supseteq Z_2 = F_0\overline{A}a$, implies $\mathcal{F}$ is not decreasing, $\mathcal{A}$ is constant on each kernel class.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{C}$. Then $C = f^{-1}D = \{a\}$, $U_C = F_0^{-1}U_D = F_0^{-1}\mathbb{Z}_4 = \mathbb{Z}_4$ and $\overline{C}a = \overline{A}a \cap F_0^{-1}\overline{D}b = Z_2 \cap \mathbb{Z}_4 = Z_2$.
Let $\mathcal{F}\mathcal{C} = \mathcal{E}$. Then $E = fC = \{b\} = D$, $U_E = F_0U_C = F_0\mathbb{Z}_4 = \mathbb{Z}_4 = U_D$ and $\overline{E}b = \overline{B}b \cap (\cup_{c\in f^{-1}fe\cap C} F_0\overline{C}c) = \mathbb{Z}_4 \cap Z_2 = Z_2 \subsetneq \mathbb{Z}_4 = \overline{D}b$, implying $\mathcal{E} \subsetneq \mathcal{D}$.

The following example shows that the above Lemma is not true if $\mathcal{A}$ is not constant on each kernel class.
Example 26: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a_1, a_2\}, \{(a_1, 2\mathbb{Z}), (a_2, 3\mathbb{Z})\}, P(\mathbb{Z}))$, $\mathcal{B}=(\{b\}, \{(b, \mathbb{Z})\}, P(\mathbb{Z}))=\mathcal{D}$, $f=\{(a_1, b), (a_2, b)\}$ and F_0 be the identity map on $\mathbb{Z}$.
Clearly, $\mathcal{A}$ is not constant each kernel class, $\overline{A}a_1 = 2\mathbb{Z}$ and $\overline{A}a_2 = 3\mathbb{Z}$.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{C}$. Then $C = f^{-1}D = \{a_1, a_2\}$, $U_C = F_0^{-1}U_D = \mathbb{Z}$ and $\overline{C}a_1 = \overline{A}a_1 \cap F_0^{-1}\overline{D}fa_1 = 2\mathbb{Z} \cap \mathbb{Z} = 2\mathbb{Z}$, $\overline{C}a_2 = \overline{A}a_2 \cap F_0^{-1}\overline{D}fa_2 = 3\mathbb{Z} \cap \mathbb{Z} = 3\mathbb{Z}$.
Let $\mathcal{F}\mathcal{C} = \mathcal{E}$. Then $E = fC = \{b\} = D$, $U_E = F_0U_C = \mathbb{Z} = U_D$ and $\overline{E}b = \overline{B}b \cap (\cup_{c\in f^{-1}fe\cap C} F_0\overline{C}c) = \overline{B}b \cap (F_0\overline{C}a_1 \cup F_0\overline{C}a_2) = \mathbb{Z} \cap (2\mathbb{Z} \cup 3\mathbb{Z}) = 2\mathbb{Z} \cup 3\mathbb{Z}$.
Clearly, $\mathcal{E}$ is not an even an s-subgroup of $\mathcal{D}$.

Lemma 4.16 For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups such that $\mathcal{A}$ is constant on each kernel class and for any s-(normal) subgroup $\mathcal{D}$ of $\mathcal{B}$, $\mathcal{F}\mathcal{F}^{-1}\mathcal{D}$ is an s-(normal) subgroup of $\mathcal{F}\mathcal{F}^{-1}\mathcal{B}$.
Proof: Since $\mathcal{D}$ is an s-(normal) subgroup of $\mathcal{B}$, by 4.11 we have $\mathcal{F}^{-1}\mathcal{D}$ is an s-(normal) subgroup of $\mathcal{F}^{-1}\mathcal{B}$. Since $\mathcal{A}$ is $\mathcal{F}$-constant both $\mathcal{F}^{-1}\mathcal{D}$ and $\mathcal{F}^{-1}\mathcal{B}$ are $\mathcal{F}$-constants by 3.1 and so by 4.10 $\mathcal{F}\mathcal{F}^{-1}\mathcal{D}$ is an s-(normal) subgroup of $\mathcal{F}\mathcal{F}^{-1}\mathcal{B}$.

As mentioned earlier in 2(B)(ii), even in crisp set up for a group homomorphism $\phi: A \to B$ and a normal subgroup D of B, $\phi\phi^{-1}D$ is not necessarily a normal subgroup of D when ϕ is a homomorphism of groups, so a counter example to show that $\mathcal{F}_p\mathcal{F}_p^{-1}\mathcal{D}$ is not necessarily an s-normal subgroup of $\mathcal{D}$, can easily be constructed using empty parameter set s-(sub) group and the details as follows:
Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\phi, \phi, P(H_2))$, $\mathcal{B} = (\phi, \phi, P(A_4)) = \mathcal{D}$, $f = \{\phi\}$ and F_0 be the inclusion homomorphism of H_2 into A_4, where $H_2 = \{e, (12)(34)\}$.
Then $\mathcal{F}$ is trivially preserving, $\mathcal{C} = \mathcal{F}^{-1}\mathcal{D} = (\phi, \phi, H_2)$ and $\mathcal{E} = \mathcal{F}\mathcal{C} = (\phi, \phi, H_2)$, clearly $\mathcal{E}$ is not normal subgroup of $\mathcal{D}$.

In what follows we show that the soft analogue of the crisp result, for subgroups D, $\phi\phi^{-1}D = D$ when ϕ is an epimorphism of groups, is true.

Lemma 4.17 For any s-epimorphism $\mathcal{F}_d: \mathcal{A} \to \mathcal{B}$ of s-groups and for any s-(normal) subgroup $\mathcal{D}$ of $\mathcal{B}$, we have $\mathcal{F}_d\mathcal{F}_d^{-1}\mathcal{D} = \mathcal{D}$.
Proof: Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{C}$. Then $C = f^{-1}D$, $U_C = F_0^{-1}U_D$ and $\overline{C}c = \overline{A}c \cap F_0^{-1}\overline{D}fc$ for all $c \in C$.
Let $\mathcal{F}\mathcal{C} = \mathcal{E}$. Then $E = fC$, $U_E = F_0U_C$ and $\overline{E}e = \overline{B}e \cap (\cup_{c\in f^{-1}e\cap C} F_0\overline{C}c)$ for all $e \in E$.
We show that $\mathcal{D} = \mathcal{E}$ or (i) $E = D$ (ii) $U_E = U_D$ and (iii) $\overline{E}e = \overline{D}e$ for all $e \in E$.
(i): $E = fC = ff^{-1}D = D$, where the last equality is due to f is onto.
(ii): $U_E = F_0U_C = F_0(F_0^{-1}U_D) = U_D$, where the last equality is due to F_0 is onto.
(iii): Let $e \in E = D$ be fixed and $c \in f^{-1}e \cap C$. Then $e = fc$, $\overline{E}e = \overline{B}e \cap (\cup_{c\in f^{-1}e\cap C} F_0\overline{C}c)$. Now we show that $F_0\overline{C}c = \overline{D}e$ for all $c \in f^{-1}e \cap C$.
(a) Since $c \in f^{-1}e$, $fc = e$. Since $\mathcal{D} \subseteq \mathcal{B}$, $\mathcal{F}$ is decreasing we have $\overline{D}e = \overline{D}fc \subseteq \overline{B}fc \subseteq F_0\overline{A}c$ which implies $\overline{D}fc \cap F_0\overline{A}c = \overline{D}fc$.
Now $F_0\overline{C}c = F_0(\overline{A}c \cap F_0^{-1}\overline{D}fc) \subseteq F_0\overline{A}c \cap F_0F_0^{-1}\overline{D}fc = F_0\overline{A}c \cap \overline{D}fc = \overline{D}fc = \overline{D}e$ which implies $F_0\overline{C}c \subseteq \overline{D}e$.
(b) Since F_0 is onto, we have $F_0F_0^{-1}\overline{D}e = \overline{D}e = \overline{D}e \cap F_0\overline{A}c$ which implies $\overline{D}e = F_0F_0^{-1}\overline{D}e \cap F_0\overline{A}c$. Let $\beta \in \overline{D}e = F_0F_0^{-1}\overline{D}e \cap F_0\overline{A}c$ which implies $\beta = F_0\alpha$, $\alpha \in F_0^{-1}\overline{D}e$, $\beta = F_0\gamma$, $\gamma \in \overline{A}c$ which implies $F_0\alpha = F_0\gamma$ which implies $\alpha - \gamma \in \ker F_0 = F_0^{-1}(0) \subseteq F_0^{-1}\overline{D}e$ with $\alpha \in F_0^{-1}\overline{D}e$ which implies $\gamma = \gamma - \alpha + \alpha \in F_0^{-1}\overline{D}e$ which implies $\gamma \in F_0^{-1}\overline{D}e \cap \overline{A}c = \overline{C}c$ which implies $\beta = F_0\gamma \in F_0\overline{C}c$ which in turn implies $\overline{D}e \subseteq F_0\overline{C}c$.
From (a) and (b) we get $F_0\overline{C}c = \overline{D}e$ for all $c \in f^{-1}e \cap C$.
Therefore, $\cup_{c\in f^{-1}e\cap C} F_0\overline{C}c = \cup_{c\in f^{-1}e\cap C} \overline{D}e = \overline{D}e$, implying $\overline{E}e = \overline{B}e \cap \overline{D}e = \overline{D}e$ or $\mathcal{E} = \mathcal{D}$.

The following example shows that the above Lemma is not true if $\mathcal{F}$ is not decreasing but $\mathcal{F}$ is onto.
Example 27: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be the s-homomorphism given by: $\mathcal{A} = (\{a\}, \{(a, Z_2)\}, P(\mathbb{Z}_4))$, $\mathcal{B} = (\{b\}, \{(b, \mathbb{Z}_4)\}, P(\mathbb{Z}_4)) = \mathcal{D}$, $f = \{(a, b)\}$ and F_0 be the identity map, where $Z_2 = \{\overline{0}, \overline{2}\}$.
Then $\mathcal{D}$ is an s-(normal) subgroup of $\mathcal{B}$, $\overline{B}fa = \overline{B}b = \mathbb{Z}_4 \supseteq \mathbb{Z}_2 = F_0\overline{A}a$, implies $\mathcal{F}$ is not decreasing.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{C}$. Then $C = f^{-1}D = \{a\}$, $U_C = F_0^{-1}U_D = F_0^{-1}\mathbb{Z}_4 = \mathbb{Z}_4$ and $\overline{C}a = \overline{A}a \cap F_0^{-1}\overline{D}b = Z_2 \cap \mathbb{Z}_4 = Z_2$.
Let $\mathcal{F}\mathcal{C} = \mathcal{E}$. Then $E = fC = \{b\} = D$, $U_E = F_0U_C = F_0\mathbb{Z}_4 = \mathbb{Z}_4 = U_D$ and $\overline{E}b = \overline{B}b \cap (\cup_{c\in f^{-1}fe\cap C} F_0\overline{C}c) = \mathbb{Z}_4 \cap Z_2 = Z_2 \neq \mathbb{Z}_4 = \overline{D}b$, implying $\mathcal{D} \neq \mathcal{E}$ or $\mathcal{F}\mathcal{F}^{-1}\mathcal{D} \neq \mathcal{D}$.

The following example shows that the above Lemma is not true if $\mathcal{F}$ is decreasing, f is onto, F_0 is not onto.
Example 28: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism defined by: $\mathcal{A} = (\{a\}, \{(a, \mathbb{Z}_2)\}, P(\mathbb{Z}_2))$, $\mathcal{B} = (\{b\}, \{b, (\overline{0})\}, P(\mathbb{Z}_4)) = \mathcal{D}$, $f = \{(a, b)\}$, $F_0 = \{(\overline{0}, \overline{0}), (\overline{1}, \overline{2})\}$, not onto and $Z_2 = \{\overline{0}, \overline{2}\}$.
Then $\mathcal{D}$ is an s-(normal) subgroup of $\mathcal{B}$, $\overline{B}fa = \overline{B}b = (\overline{0}) \subseteq Z_2 = F_0\overline{A}a$, implies $\mathcal{F}$ is decreasing.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{C}$. Then $C = f^{-1}D = \{a\}$, $U_C = F_0^{-1}U_D = F_0^{-1}\mathbb{Z}_4 = \mathbb{Z}_2$ and $\overline{C}a = \overline{A}a \cap F_0^{-1}\overline{D}fa = \mathbb{Z}_2 \cap F_0^{-1}(\overline{0}) = \mathbb{Z}_2 \cap (\overline{0}) = (\overline{0})$
Let $\mathcal{F}\mathcal{C} = \mathcal{E}$. Then $E = fC = fa = \{b\} = D$, $U_E = F_0U_C = F_0\mathbb{Z}_2 = Z_2 \neq = \mathbb{Z}_4 = U_D$, implying $\mathcal{E} \neq \mathcal{D}$ or $\mathcal{F}\mathcal{F}^{-1}\mathcal{D} \neq \mathcal{D}$.

The following example shows that the above Lemma is not true if $\mathcal{F}$ is decreasing, f is not onto, F_0 is onto.
Example 29: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism defined by: $\mathcal{A} = (\{a_1, a_2\}, \{(a_1, Z_2), (a_2, Z_2)\}, P(\mathbb{Z}_4))$, $\mathcal{B} = (\{(b_1, b_2)\}, \{(b_1, (\overline{0})), (b_2, Z_2)\}, P(\mathbb{Z}_4)) = \mathcal{D}$, $f = \{(a_1, b_1), (a_2, b_1)\}$, F_0 be the identity map on $\mathbb{Z}_4$, $Z_2 = \{\overline{0}, \overline{2}\}$.
Then $\mathcal{D}$ is an s-(normal) subgroup of $\mathcal{B}$, f is not onto, $\overline{B}fa_1 = \overline{B}b_1 = (0) \subseteq Z_2 = F_0\overline{A}a_1$ and $\overline{B}fa_2 = \overline{B}b_1 = (0) \subseteq Z_2 = F_0\overline{A}a_2$, implies $\mathcal{F}$ is decreasing.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{C}$. Then $C = f^{-1}D = f^{-1}b_1 = \{a_1, a_2\}$, $U_C = F_0^{-1}U_D = F_0^{-1}\mathbb{Z}_4 = \mathbb{Z}_4$ and $\overline{C}a_1 = \overline{A}a_1 \cap F_0^{-1}\overline{D}fa_1 = Z_2 \cap (0) = (0) = \overline{C}a_2$.
Let $\mathcal{F}\mathcal{C} = \mathcal{E}$. Then $E = fC = f(a_1, a_2) = \{b_1\} \neq \{(b_1, b_2)\} = D$, implying $\mathcal{E} \neq \mathcal{D}$ or $\mathcal{F}\mathcal{F}^{-1}\mathcal{D} \neq \mathcal{D}$.

CONCLUSION

In this paper we introduced the notions of generalized soft homomorphism of generalized soft groups, generalized soft (inverse) image of generalized soft (normal) subgroup under generalized soft homomorphism, generalized soft kernel of a generalized soft homomorphism etc., generalizing the corresponding existing notions for a soft group over a group and showed that several of the crisp theoretic results naturally extended to these new objects too.

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