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The Homotopy Perturbation Method for Ordinary Differential Equation Method

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ABSTRACT

In this work, we adopted the theory of ordinary differential equations in the derivation of the model. The effective reproduction number of the model is determined and stability equilibria analyzed. The disease free equilibrium point of the model is shown to be locally asymptotically stable when its corresponding effective reproduction number is less than unity. Using the Lyapunov function of the Goh-Volterra type, we also proved that the endemic equilibrium point is globally asymptotically stable when its corresponding reproduction number is greater than unity. The derived model equations in first order ordinary differential equations were transformed into the He's Homotopy Perturbation Method, HPM.

Keywords: *Homotopy Perturbation Method, Lyapunov Function, stability analysis.*

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I. INTRODUCTION

In recent years, the application of the Homotopy Perturbation Method (HPM) in non-linear problems has been developed by scientists and engineers, because this method continuously deforms the difficult problem under study into a simple problem which is easier to solve. The homotopy perturbation method was proposed first by He in, [1] and was developed and improved by [2] , [3], and [4]. Homotopy, is an important part of differential topology. Actually the HPM is a coupling of the traditional perturbation method and the homotopy method in topology [3]. The homotopy perturbation method (HPM) provides an approximate analytical solution in a series form. HPM has been widely used by numerous researchers successfully for different physical systems such as, bifurcation, asymptotology, nonlinear wave equations, oscillators with discontinuities by [5],[6], reaction-duffision equation and heat radiation equation by [7];[8].

II. MODEL FORMULATION AND ASSUMPTIONS

The model is being developed based on the following assumptions;

a. The virus/disease, HIV/AIDS kills
b. An individual who contact this disease will surely die of it if not treated
c. Vertical transmission is not considered;
d. There is no control measure with 100% efficacy to prevent HIV/AIDS;
e. The campaign reduces the rate of transmission because those who are properly informed will reduce their exposure to the infection whenever they meet any infectious opportunity;
f. The population is homogeneous
g. Those in the restricted class can still engage in sexual activities at the slightest opportunity

Model Formulation and Procedures

The flow transmission parameters and variables in the model building blocks are as follows; the recruitment into susceptible unvaccinated class, $S_u(t)$ by the parameter, α. This class is divided into two components; the susceptible unvaccinated and restricted, $S_{ur}(t)$, and susceptible unvaccinated that use condom $S_{uc}(t)$. These populations are recruited by the parameter, σ *and* $(1-\sigma)$ respectively. A proportion, g of the population, $S_{ur}(t)$ leaves into the susceptible unvaccinated who are restricted and also use condom, $S_{ucr}(t)$ and $(1-f_2)\lambda$ proportion of $S_{ur}(t)$ is recruited into $S_{ue}(t)$, certain fraction, h of the class, $S_{uc}(t)$ moves to the $S_{ucr}(t)$. A proportion leaves by $(1-f_1)\lambda$ of this class goes into the $S_{ue}(t)$ class. Another fraction, $\lambda(1-f)$ leaves the unvaccinated, restricted class that uses condom for exposed class. Fraction recruited into the infected class, $I(t)$ by ρ_2 from $S_{ue}(t)$ class. Population of the full blown AIDS class is recruited from the infected class at a rate, γ. There is the disease induced death rate, d_1 in the full blown AIDS class. The parameter. c is the campaign for the usage of condom and the restriction of free movement. The force of infection is denoted by . There is natural death rate, μ in all the compartments.

The following are the corresponding model equations resulting from the assumptions

$$\left.\begin{aligned}
\frac{dS_u}{dt} &= \alpha - \sigma c S_u - (1-\sigma)cS_u - \mu S_u \\
\frac{dS_{uc}}{dt} &= \sigma c S_u - (1-f_1)\lambda S_{uc} - hS_{uc} - \mu S_{uc} \\
\frac{dS_{ur}}{dt} &= (1-\sigma)cS_u - gS_{ur} - (1-f_2)\lambda S_{ur} - \mu S_{ur} \qquad (1) \\
\frac{dS_{ucr}}{dt} &= hS_{uc} + gS_{ur} - (1-f)\lambda S_{ucr} - \mu S_{ucr} \\
\frac{dS_{ue}}{dt} &= (1-f_1)\lambda S_{uc} + (1-f)\lambda S_{ucr} + (1-f_2)\lambda S_{ur} - \rho_1 S_{ue} - \mu S_{ue} \\
\frac{dI}{dt} &= \rho_1 S_{ue} - \gamma I - \mu I \\
\frac{dA}{dt} &= \gamma I - dA - \mu A
\end{aligned}\right\}$$

$\lambda = \frac{n_i \varepsilon_i I}{N}$, effective contact rate with the following initial conditions;

$S_u(0) > 0, S_{uc}(0) > 0, S_{ucr}(0) > 0, S_{ur} > 0, S_{ue}(0) > 0, I(0) > 0, A(0) > 0$

The total population therefore given as

$N = S_u + S_{uc} + S_{ucr} + S_{ur} + S_{ue} + I + A$

That is,

$\frac{dN}{dt} = \alpha - \alpha S - \sigma c S_u - (1-\sigma)cS_u - \mu S_u$

$\Longrightarrow \alpha - \mu S_u - \mu S_{uc} - \mu S_{ur} - \mu S_{ucr} - \mu S_{ue} - \mu I - \mu A = \frac{dN}{dt}$

$\frac{dN}{dt} = \alpha - \mu N$

$\Longrightarrow \alpha - \mu N = 0$, since $\frac{dN}{dt} = 0$ at equilibrium point,

$$N = \frac{\alpha}{\mu} \qquad (2)$$

$N(t) \rightarrow \frac{\alpha}{\mu}$ at $t \rightarrow \infty$

Therefore,

$$N(0) \geq \frac{\alpha}{\mu} \qquad (3)$$

Which gives the following feasible region

$\varsigma = (S_u, S_{uc}, S_{ur}, S_{ucr}, S_{ue}, I, A) \epsilon R^7{}_+$

III. INVARIANT REGION

Theorem 1: The solutions of the model (1) are feasible for all $t > 0$ if they enter the invariant region, ς

Proof:

We let $\varsigma = (S_u, S_{uc}, S_{ur}, S_{ucr}, S_{ue}, I, A) \in R_+{}^7$ be any solution of the system, (1) with positive initial conditions. From (2) we have that in the absence of the virus, the total population will be;

$N \leq \alpha - \mu N$

Evaluating, we obtain

$\frac{d}{dt}(Ne^{\mu t}) \leq \alpha e^{\mu t}$

Integrating, we obtain the following expression and applying initial conditions, we obtain

$$N \leq \frac{\alpha}{\mu} + (N_0 - \frac{\alpha}{\mu})e^{-\mu t} \qquad (4)$$

Applying the inequality theorem by [9] on differential equations yields the following results

$0 \leq N \leq \frac{\alpha}{\mu}$ as $t \rightarrow \infty$

We shall let $p_3 = \frac{\alpha}{\mu}$, where $Kp_3$is the carrying capacity as the total population approaches Kp_3, therefore, the feasible solution of the model enters the region,

$\varsigma = \left\{(S_u(t), S_{uc}(t), S_{ur}(t), S_{ucr}(t), S_{ue}(t), I(t), A(t)) \epsilon R^7{}_+ \leq \frac{\alpha}{\mu}\right\}$

And is positively- invariant and attracting for the model system (1). This completes the proof.

Positivity of Solution

Theorem 2: we shall show that the model without vaccination and treatment in the presence of condom usage and restriction of free movement is positive at all time, $t > 0$.

Let
$\varsigma = \left\{ S_u(0), S_{uc}(0), S_{ur}(0), S_{ucr}(0) S_{ue}(0), I(0), A(0) \geq 0 \quad \epsilon R^7{}_+ \right\}$
the solution set; $S_u(t), S_{uc}(t), S_{ur}(t), S_{ucr}(t) S_{ue}(t), I(t), A(t)$ of the systems of equation (2) is positive for all $t > 0$

Proof:

From the first of (1) we have that;

$\frac{dS_u}{dt} = \alpha - \sigma c S_u - (1-\sigma) c S_u - \mu S_u$

We have that;

$\frac{dS_u}{dt} = \alpha - \sigma c S_u - (1-\sigma) c S_u - \mu S_u \geq -[\sigma c + (1-\sigma)c + \mu] S_u$

$\Rightarrow \frac{dS_u}{dt} \geq -[\sigma c + (1-\sigma)c + \mu] S_u$

$\frac{dS_u}{S} \geq -[\sigma c + (1-\sigma)c + \mu] S_u dt$

Integrating;

$\ln S_u \geq -[\sigma c + (1-\sigma)c + \mu] t + C$

Where C is the constant of integration, applying initial condition is gives

$S_u(t) \geq S_u(0) e^{-(\sigma c + (1-\sigma)c + \mu)t} \geq 0$, $S_{uc}(t) \geq S_{uc}(0) e^{-((1-f_1)\lambda + h + \mu)t} \geq 0$ (5)

Since

$\left((1-f_1)\lambda + h + \mu\right) \geq 0$,

Similarly, it can be shown that;

$$\left.\begin{aligned}
&S_{uc}(t) \geq S_{uc}(0) e^{-(h + (1-f_1)\lambda + \mu)t} \geq 0\\
&S_{ur}(t) \geq S_{ur}(0) e^{-((g + (1-f_2)\lambda + \mu))t} \geq 0\\
&S_{ucr}(t) \geq S_{ucr}(0) e^{-((1-f)\lambda + \mu)t} \geq 0\\
&S_{ue}(t) \geq S_{ue}(0) e^{-(\rho_1 + \mu)t} \geq 0\\
&I(t) \geq I(0) e^{-(\gamma + \mu)t} \geq 0\\
&A(t) \geq A(0) e^{-(d + \mu)t} \geq 0
\end{aligned}\right\} \qquad (6)$$

We proved that all the solutions of the systems of equation of (1) are non-negative for all$t \geq 0$. The above theorem is important because it guarantees that the model is well posed and biologically feasible in the region ς since population cannot be negative. Thus for any starting non-negative initial conditions, the trajectory lies in ς. Therefore, the system is both mathematically and epidemiologically well-posed, [10]. Thus it is significant to consider the dynamics of the flow by the model (1) in ς.

Equilibrium Points and Stability Analysis

The model system (1) has two non-negative equilibria; Disease free equilibrium (DFE) point and endemic equilibrium point.

The Disease Free Equilibrium Point and the effective reproduction number R_e

Considering the feasible region,$\varsigma = (S_u, S_{uc}, S_{ur}, S_{ucr}, S_{ue}, I, A) \epsilon R^7{}_+$

The DFE point is obtained by setting the right hand side of (1) to zero which means that, in the absence of the disease, we have that;

$S_{uc} = S_{ur} = S_{ucr} = S_{ue} = I = A = 0$, therefore, the above equation will now reduce to;

$S_u = \frac{\alpha}{\mu}$ (7)

Hence, $\varepsilon_0 = (S_u, S_{uc}, S_{ur}, S_{ucr}, S_{ue}, I, A) = (\frac{\alpha}{\mu}, 0,0,0,0,0,0)$. This is the DFE.

The local stability of E can be analyzed by the threshold R_e called the reproduction number which is defined as the average number of secondary infections generated by a single infected individual in a totally susceptible population. This is determined using the next generation method on system (1) as used in [11] in the form of matrices F (non-negative) and V (non-singular). Where F denote infection terms and V the transition term at E. It follows that the effective reproduction number $R_e = \rho(FV^{-1})$ of the model system eq. (1) is obtained to be

$R_e = \frac{\{(1-f_1)+(1-f)+(1-f_2)\} n_1 \varepsilon_1}{(\rho_1+\mu)} + \frac{\{(1-f_1)+(1-f)+(1-f_2)\} n_2 \varepsilon_2 \rho_1}{(\rho_1+\mu)(\gamma+\mu)} + \frac{\{(1-f_1)+(1-f)+(1-f_2)\} \rho_1 \gamma n_3 \varepsilon_3}{(\rho_1+\mu)(\gamma+\mu)(d+\mu)}$

Where ρ is the spectral radius, i.e. the dominant eigen-value of $\left|(FV^{-1}) - \lambda I\right| = 0$

The threshold quantity R_e is the effective reproduction number of the system (1) for the control of HIV/AIDS in the unvaccinated population with condom and restriction as control strategies.

The DFE of the model (1) is locally asymptotically stable if $R_e < 1$ and unstable otherwise.

The **endemic equilibrium** point of the system (1) denoted by E^* is expressed in terms of R_e as;

$s^*{}_u = \frac{\alpha b}{\{\sigma c + (1-\sigma)c + \mu\}(\alpha+\mu)}$, $s^*{}_{uc} = \frac{\sigma c \alpha b}{\{(1-f_1)\lambda + h + \mu\}\{\sigma c + (1-\sigma)c + \mu\}(\alpha+\mu)}$

$s^*_{ur} = \frac{(1-\sigma)c^2\sigma\alpha b}{\{g+(1-f_2)\lambda+\mu\}\{(1-f_1)\lambda+h+\mu\}\{\sigma c+(1-\sigma)c+\mu\}(\alpha+\mu)}$

$s^*_{ucr} = \frac{[g(1-\sigma)c+h\{g+(1-f_2)\lambda+\mu\}]\sigma c\alpha b}{\{(1-f)\lambda+\mu\}\{g+(1-f_2)\lambda+\mu\}\{(1-f_1)\lambda+h+\mu\}\{\sigma c+(1-\sigma)c+\mu\}(\alpha+\mu)}$

$s^*_{ue} =$
$\sigma c\alpha b[(1-\sigma)(1-f)gc\lambda + \lambda(1-f)\{g+(1-f_2)\lambda+\mu\} + \lambda c(1-\sigma)(1-f_2)\{(1-f)\lambda+\mu\} + \{g+(1-f_2)\lambda+\mu\}\{(1-f)\lambda+\mu\}(1-f_1)\lambda]$
$(\rho_1+\mu)\{(1-f)\lambda+\mu\}\{g+(1-f_2)\lambda+\mu\}\{(1-f_1)\lambda+h+\mu\}\{\sigma c+(1-\sigma)c+\mu\}(\alpha+\mu)$

$a^* = \frac{\gamma i^*}{(d+\mu)}, \quad i^* = \frac{\rho_1 s^*_{ue}}{(\gamma+\mu)}$

So the endemic state exists as can be observed from above and is therefore greater than unity.

Global Stability of the Endemic Equilibrium, E^*

Theorem (3). if $R_e > 1$ then the endemic equilibrium of the model (1) is globally and asymptotically stable.

Proof:

We construct the lyapunov function to establish the global stability of the endemic equilibrium, $\boldsymbol{E}^*$ of the system as follows;

$L(s^*, s^*_u, s^*_{uc} s^*_{ur}, s^*_{ucr}, s^*_{ue}, i^*, a^*) =$

$$(s - s^* - s^*\log\frac{s^*}{s}) + (s_u - s^*_u - s^*_u\log\frac{s^*_u}{s_u}) + (s_{uc} - s^*_{uc} - s^*_{uc}\log\frac{s^*_{uc}}{s_{uc}}) + (s_{ur} - s^*_{ur} - s^*_{ur}\log\frac{s^*_{ur}}{s_{ur}}) + (s_{ucr} - s^*_{ucr} - s^*_{ucr}\log\frac{s^*_{ucr}}{s_{ucr}}) + (s_{ue} - s^*_{ue} - s^*_{ue}\log\frac{s^*_{ue}}{s_{ue}}) +$$

$(i - i^* - i^*\log\frac{i^*}{i}) + (a - a^* - a^*\log\frac{a^*}{a})$ (8)

Evaluating the derivatives of L directly along the solution path of (8) and collecting the negative and the positive terms, we obtain the following;

$\frac{dL}{dt} = X - Y$ (9)

Where;

$X = \frac{(s+s^*)}{s}b + \frac{(s_u+s^*_u)}{s_u}\alpha(s+s^*) + \frac{(s_{uc}+s^*_{uc})}{s_{uc}}\sigma c(s_u+s^*_u) + \frac{(s_{ur}+s^*_{ur})}{s_{ur}}(1-\sigma)c(s_u+s^*_u) + \frac{(s_{ucr}+s^*_{ucr})}{s_{ucr}}\{h(s_{uc} +$
suc∗+gsur+sur∗}+(sue+sue∗)sue1−f1λsuc+suc∗+1−fλ(sucr+sucr∗+1−f2λsur+sur∗}

$+(1-f_2)\lambda(s_{ur}+s^*_{ur})\}\} + \frac{(i+i^*)}{i}\rho_1(s_{ue}+s^*_{ue}) + \frac{(a+a^*)}{a}\gamma(i+i^*)$

$Y = -(\alpha+\mu)\frac{(s+s^*)^2}{s} - (\sigma c+(1-\sigma)c+\mu)\frac{(s_u+s^*_u)^2}{s_u} - ((1-f_1)\lambda+h+\mu)\frac{(s_{uc}+s^*_{uc})^2}{s_{uc}} - (g+(1-f_2)\lambda+$
μsur+sur∗2sur−1−fλ+μsucr+sucr∗2sucr−ρ1+μsue+sue∗2sue−γ+μi+i∗2i−d+μa+a∗2a

This follows from (9) that;

If $X < Y$ then, $\frac{dL}{dt}$ will be negative definite that is $\frac{dL}{dt} < 0$ and $\frac{dL}{dt} = 0$

Only when;

$s = s^*, s_u = s^*_u, s_{uc} = s^*_{uc}, s_{ur} = s^*_{ur}, s_{ucr} = s^*_{ucr}, s_{ue} = s^*_{ue}, i = i^*, a = a^*$

Therefore, the largest compact invariant set,

$\{s^*_u, s^*_{uc}, s^*_{ur}, s^*_{ucr}, s^*_{ue}, i^*, a^* \epsilon \Omega_3 : \frac{dL}{dt} = 0\}$ is the only singleton set E^* where E^* *is* globally asymptotically stable in Ω_3 if $X < Y$.

Solution of the model using Homotopy Perturbation Method

To demonstrate the basic idea of He's homotopy perturbation method, we consider the non linear differential equation, [2].

$A(u) - f(r) = 0 \qquad r \in \Omega$ (10)

Subject to the boundary condition of:

$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma$ (11)

Where;

A is the general differential operator,

B is the boundary operator

$f(r)$; a known analytical solution and

Γ is the boundary of the domain Ω , [12].

The general operator, A can be divided into two parts namely; L and N where L is linear part and N is the non linear part. Hence (10) can therefore be written as;

$L(u) + N(u) - f(r) = 0 \qquad r \in \Omega$ (12)

We shall now construct a homotopy $V(r, p)$ such that

$V(r, p): \Omega \times [0,1] \to R$ which satisfies

$H(r, p) = (1 - p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0$ (13)

$P \in [0,1], \quad r \in \Omega$

OR

$H(r, p) = L(v) - L(u_0) + pL(u_0) + [N(v) - f(r)] = 0$ (14)

Where

$L(u)$ is the linear part

$L(u) = L(v) - L(u_0) + pL(u_0)$ and

$N(u)$ is the non-linear part

$N(u) = pN(v)$

$P \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of equation (10) which satisfies the boundary conditions.

Obviously, considering equations(13) and (14), we have

$H(v, 0) = L(v) - L(u_0) = 0$ (15)

$H(v, 1) = A(v) - f(r) = 0$ (16)

The changing process of p from zero to unity is just that of $V(r, p)$ from u_0 to $u(r)$. In topology, this is called deformation while $L(v) - L(u_0)$, $A(v) - f(r)$ are called homotopy.

According to Homotopy perturbation method (HPM), we can first use the embedding parameter, p as a small parameter and assume solution for equation (13) and (14) which can be expressed as;

$V = v_0 + pv_1 + p^2 v_2 + \cdots$ (17)

If we let $p = 1$, we will obtain an approximate solution of equation (17) as

$U = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots$ (18)

Equation (18) is the analytical solution of (10) by homotopy perturbation method.

He (2003), (2006) makes the following suggestion for convergence of equation (18)

(1). The second derivative of $N(v)$ with respect to V must be small because parameter, p must be relatively large i.e $p \to 1$

(2). The norm of $L^{-1} \frac{\partial N}{\partial V}$ must be smaller than one so that the series converge.

Considering the following systems of non-linear ordinary differential equations in (1)

We let,

$k_2 = (1 - \sigma)c, \quad k_3 = (k_2 + \sigma c + \mu), \quad k_4 = (1 - f_1)\lambda,$

$k_5 = (k_4 + h + \mu), k_6 = (1 - f_2)\lambda, \quad k_7 = (k_6 + g + \mu),$

$k_8 = (1 - f)\lambda, k_9 = (k_8 + \mu), \quad k_{10} = (\rho_1 + \mu), k_{11} = (\gamma + \mu), k_{12} = (d + \mu)$

Rewriting (1) in a more compact form we get

$$\left.\begin{aligned}
\frac{dS_u}{dt} &= \alpha S - k_3 S_u \\
\frac{dS_{uc}}{dt} &= \sigma c S_u - k_5 S_{uc} \\
\frac{dS_{ur}}{dt} &= k_2 S_u - k_7 S_{ur} \\
\frac{dS_{ucr}}{dt} &= h S_{uc} + g S_{ur} - k_9 S_{ucr} \\
\frac{dS_{ue}}{dt} &= k_4 S_{uc} + k_8 S_{ucr} + k_6 S_{ur} - k_{10} S_{ue} \\
\frac{dI}{dt} &= \rho_1 S_{ue} - k_{11} I \\
\frac{dA}{dt} &= \gamma I - k_{12} A
\end{aligned}\right\} \quad (19)$$

We now apply HPM on the system (19) by assuming the solution as;

$$\left.\begin{aligned}
S_u &= r_0 + Pr_1 + P^2 r_2 + \cdots \\
S_{uc} &= t_0 + Pt_1 + P^2 t_2 + \cdots \\
S_{ur} &= u_0 + Pu_1 + p^2 y_2 + \cdots \\
S_{ucr} &= v_0 + Pv_1 + P^2 v_2 + \cdots \\
S_{ue} &= w_0 + Pw_1 + P^2 w_2 + \cdots \\
I &= x_0 + Px_1 + P^2 x_2 + \cdots \\
A &= y_0 + Py_1 + P^2 y_2 + \cdots
\end{aligned}\right\} \quad (20)$$

From the first equation of(19),

$\frac{dS_u}{dt} = \alpha - k_3 S_u$,

The linear part is

$\frac{dS_u}{dt} = 0$

and the non-linear part is

$\alpha - k_3 S_u$

We now apply HPM

$\Rightarrow (1-P)\frac{dS_u}{dt} + P\left[\frac{dS_u}{dt} - \alpha + k_3 S_u\right] = 0$

Expanding, this gives

$\frac{dS_u}{dt} - P\frac{dS_u}{dt} + P\frac{dS_u}{dt} - P(\alpha - k_3 S_u) = 0$

$\Rightarrow \frac{dS_u}{dt} - P(\alpha - k_3 S_u) = 0$

$\Rightarrow \frac{dS_u}{dt} - P\alpha + Pk_3 S_u = 0 \qquad (21)$

Substituting the first and second equations of (20) into (21) gives

$(r_0' + Pr_1' + P^2 r_2' + \cdots +) - P\alpha + Pk_3(r_0 + Pr_1 + P^2 r_2 + \cdots)$

Collecting the coefficient of powers of P, we have;

$$\left.\begin{aligned} P^0&: r_0' = 0 \\ P^1&: r_1' - \alpha + k_3 r_0 = 0 \\ P^2&: r_2' - \alpha + k_3 r_1 = 0 \end{aligned}\right\} \qquad (22)$$

Similarly we have;

$$\left.\begin{aligned} P^0&: t_0' = 0 \\ P^1&: t_1' - \sigma c r_0 + k_5 t_0 = 0 \\ P^2&: t_2' - \sigma c r_1 + k_5 t_1 = 0 \end{aligned}\right\} \qquad (23)$$

$$\left.\begin{aligned} P^0&: u_0' = 0 \\ P^1&: u_1' - k_2 r_0 + k_7 u_0 = 0 \\ P^2&: u_2' - k_2 r_1 + k_7 u_1 = 0 \end{aligned}\right\} \qquad (24)$$

$$\left.\begin{aligned} P^0&: v_0' = 0 \\ P^1&: v_1' - ht_0 - gu_0 + k_9 v_0 = 0 \\ P^2&: v_2' - ht_1 - gu_1 + k_9 v_1 = 0 \end{aligned}\right\} \qquad (25)$$

$$\left.\begin{aligned} P^0&: w_0' = 0 \\ P^1&: w_1' - k_4 t_0 - k_6 u_0 - k_8 v_0 + k_{10} w_0 = 0 \\ P^2&: w_2' - k_4 t_1 - k_6 u_1 - k_8 v_1 + k_{10} w_1 = 0 \end{aligned}\right\} \qquad (26)$$

$$\left.\begin{aligned} P^0&: x_0' = 0 \\ P^1&: x_1' - \rho_2 w_0 + k_{11} x_0 = 0 \\ P^2&: x_2' - \rho_2 w_1 + k_{11} x_1 = 0 \end{aligned}\right\} \qquad (27)$$

$$\left.\begin{aligned} \mathrm{P}^0&: \mathrm{y}_0' = 0 \\ \mathrm{P}^1&: \mathrm{y}_1' - \gamma \mathrm{x}_0 + \mathrm{k}_{12} \mathrm{y}_0 = 0 \\ \mathrm{P}^2&: \mathrm{y}_2' - \gamma \mathrm{x}_1 + \mathrm{k}_{12} \mathrm{y}_1 = 0 \end{aligned}\right\} \qquad (28)$$

From the first equation of (22),

$r_0' = 0$

$\frac{dr_0}{dt} = 0$

$\Rightarrow dr_0 = 0$

Integrating gives us

$\int dr_0 = S_{u0}$

$\therefore r_0 = d_1$

Where d_1 is constant of integration. Applying the initial condition we have

$r_0(0) = S_{u0}$

$\Rightarrow d_1 = S_{u0}$

$\therefore r_0 = S_{u0} \qquad (29)$

Similarly, we have that;

$\therefore t_0 = S_{uc0} \qquad (30)$

$\therefore u_0 = S_{ur0} \qquad (31)$

$\therefore v_0 = S_{ucr0}$ (32)
$\therefore w_0 = S_{ue0}$ (33)
$\therefore x_0 = I_0$ (34)
$\therefore y_0 = A_0$ (35)
From the second equation of (22),
$r_1' - \alpha + k_3 r_0 = 0,$
$r_1' = \alpha - k_3 r_0$
$\Rightarrow \frac{dr_1}{dt} = \alpha - k_3 r_0$
$\Rightarrow dr_1 = (\alpha - k_3 r_0)dt$ (36)
Substituting (29) into (36) we obtain;
$dr_1 = (\alpha - k_3 S_{u0})dt$
Integrating with respect to t, we have;
$r_1 = (\alpha - k_3 S_{u0})t + d_9$
Where d_9 is constant of integration. Applying the initial condition we have;
$r_1(0) = 0, \ \Rightarrow d_9 = 0$
$\therefore r_1 = (\alpha - k_3 S_{u0})t$ (37)
Similarly,
$\therefore t_1 = (\sigma c S_{u0} - k_5 S_{uc0})t$ (38)
$\therefore u_1 = (k_2 S_{u0} - k_7 S_{ur0})t$ (39)
$\therefore v_1 = (hS_{uc0} + gS_{ur0} - k_9 S_{ucr0})t$ (40)
$\therefore w_1 = (k_4 S_{uc0} + k_6 S_{ur0} + k_8 S_{ucr0} - k_{10} S_{ue0})t$ (41)
$\therefore x_1 = (\rho_2 S_{ue0} - k_{11} I_0)t$ (42)
$\therefore y_1 = (\gamma I_0 - k_{12} A_0)t$ (43)
From the third equation of (22),
$r_2' - \alpha + k_3 r_1 = 0$
$r_2' = \alpha - k_3 r_1$
$\Rightarrow \frac{dr_2}{dt} = \alpha - k_3 r_1$
$\Rightarrow dr_2 = (\alpha - k_3 r_1)\, dt$ (44)
Substituting (37) into (44) we obtain;
$dr_2 = [\alpha - k_3(\alpha - k_3 S_{u0})t]dt$
$dr_2 = [\alpha - k_3(\alpha - k_3 S_{u0})]tdt$
$dr_2 = [\alpha - \alpha k_3 + k_3^2 S_{u0}]tdt$
Integrating both sides with respect to t, we have;
$r_2 = [\alpha - \alpha k_3 + k_3^2 S_{u0}]\frac{t^2}{2} + d_{17}$
Where d_{17} is constant of integration. Applying the initial condition we have;
$r_2(0) = 0, \ \Rightarrow d_{17} = 0$
$\therefore r_2 = [\alpha - \alpha k_3 + k_3^2 S_{u0}]\frac{t^2}{2}$ (45)
Similarly,
$\therefore t_2 = [\sigma c \alpha - c\sigma k_5 S_{u0} + k_5^2 S_{uc0}]\frac{t^2}{2}$ (46)
$\therefore u_2 = [\alpha - (k_2 k_3 + k_2 k_7)S_{u0} + k_7^2 S_{ur0}]\frac{t^2}{2}$ (47)
$\therefore v_2 = [(h\sigma c - k_2 g)S_{u0} - (hk_3 + hk_9)S_{uc0} - (gk_7 + gk_9)S_{ur0} + k_9^2 S_{ucr0}]\frac{t^2}{2}$ (48)
$\therefore W_2 = [(k_4\sigma c + k_2 k_6)S_{u0} - (k_4 k_5 + k_4 k_{10} - hk_8)S_{uc0} - (k_6 k_7 + k_6 k_{10} - gk_8)S_{ur0} - (k_8 k_{10} + gk_9)S_{ucr0} + k_{10}^2 S_{ue0}]\frac{t^2}{2}$ (49)
$\therefore x_2 = [\rho_2 k_4 S_{uc0} + (\rho_2 k_6 + \rho_2 k_8)S_{ucr0} - (\rho_2 k_{10} + \rho_2 k_{11})S_{ue0} + k_{11}^2 I_0]\frac{t^2}{2}$ (50)
$\therefore y_2 = [\gamma\rho_2 S_{ue0} - (\gamma k_{11} + \gamma k_{12})I_0 + k_{12}^2 A_0]\frac{t^2}{2}$ (51)
Substituting (29) , (41) and (45) into the second equation of (20),we obtain;
$S_u(t) = S_{u0} + P(\alpha - k_3 S_{u0})t + P^2[\alpha + k_3^2 S_{u0}]\frac{t^2}{2}$ (52)
Setting $p = 1$, the solution (52) becomes;
$S_u(t) = S_{u0} + (\alpha - k_3 S_{u0})t + [\alpha + k_3^2 S_{u0}]\frac{t^2}{2} + \ldots$ (53)
Similarly,
$S_{uc}(t) = S_{uc0} + (\sigma c S_{u0} - k_5 S_{uc0})t + [\sigma c \alpha S_0 - c\sigma k_5 S_{u0} + k_5^2 S_{uc0}]\frac{t^2}{2} + \ldots$ (54)

$S_{ur}(t) = S_{ur\,0} + (k_2 S_{u0} - k_7 S_{ur\,0})t + [-(k_2 k_3 + k_2 k_7)S_{u0} + k_7^2 S_{ur\,0}]\frac{t^2}{2} + \ldots$ (55)

$S_{ucr}(t) = S_{ucr\,0} + (hS_{uc\,0} + gS_{ur\,0} - k_9 S_{ucr\,0})t + [(h\sigma c - k_2 g)S_{u0} - (hk_3 + hk_9)S_{uc\,0} - (gk_7 + gk_9)S_{ur\,0} + k_9^2 S_{ucr\,0}]\frac{t^2}{2} + \ldots$ (56)

$S_{ue}(t) = S_{ue\,0} + (k_4 S_{uc\,0} + k_6 S_{ur\,0} + k_8 S_{ucr\,0} - k_{10} S_{e0})t + [(k_4 \sigma c + k_2 k_6)S_{u0} - (k_4 k_5 + k_4 k_{10} - hk_8)S_{uc\,0} - (k_6 k_7 + k_6 k_{10} - gk_8)S_{ur\,0} - (k_8 k_{10} + gk_9)S_{ucr\,0} + k_{10}^2 S_{ue\,0}]\frac{t^2}{2} + \ldots$ (57)

$I(t) = I_0 + (\rho_2 S_{ue\,0} - k_{11} I_0)t + [\rho_2 k_4 S_{uc\,0} + (\rho_2 k_6 + \rho_2 k_8)S_{ucr\,0} - (\rho_2 k_{10} + \rho_2 k_{11})S_{ue\,0} + k_{11}^2 I_0]\frac{t^2}{2} + \ldots$ (58)

$A(t) = A_0 + (\gamma I_0 - k_{12} A_0)t + [\gamma \rho_2 S_{ue\,0} - (\gamma k_{11} + \gamma k_{12})I_0 + k_{12}^2 A_0]\frac{t^2}{2} + \ldots$ (59)

Hence equations (53) to (59) are our model equations in HPM.

IV. CONCLUSION

Through the approach presented in this paper, the HPM can be extended to solve ordinary differential equations. The method requires less work with very little cost (when compared with other numerical methods like classical RK).

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