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Hadamard Admissibility Series And Strongly Regular Graphs

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ABSTRACT
Let G be a primitive strongly regular graph and A its adjacency matrix. Let $\mathcal{A}$ *be the Euclidean Jordan subalgebra of the Euclidean Jordan algebra V of real symmetric matrices of order n, equipped with the Jordan product and with the usual inner product of two matrices of order n, this is with the inner product that is the trace of the usual product of matrices, spanned by the identity of order n and the natural powers of A. By an algebraic analysis of the spectra of a particular Hadamard series of an element of* $\mathcal{A}$ *we establish an admissibility condition on the spectra and on the parameters of a strongly regular graph.*
KEYWORDS;-*Euclidean Jordan Algebras, Graph Theory, Strongly regular graphs.*

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I. INTRODUCTION

An Euclidean Jordan algebrais an algebraic structure that is a good environment for theanalysis interior point methods, see [1], [2] and [3]. Recently, there are many applications of Euclidean Jordan algebras, to the other fields of Mathematics, namely to the generalization of the properties of symmetric matrices to Euclidean Jordan algebras were established, see [4], [5] and [6], and to combinatorics, see [7], [8], [9] ,[10] ,[11], [12], and to statistics [13].

In this paper we pretend, in the environment of Euclidean Jordan algebras, to establish inequalities of admissibility on the parameters and on the spectra of a strongly regular graph,.

The organization of the paper is as follows. Innext section II we present a brief description of the principal properties of Euclidean Jordan algebras. In the third section, we present the more relevant definitions and results about strongly regular graphs for this paper.in section IV, we deduce inequalities some inequalities on the parameters of a primitive strongly regular graph. Finally, in section V we present some numerical results and some conclusions.

II. PROPERTIES OF EUCLIDEAN JORDAN ALGEBRAS

One finds detailed textbooks about Euclidean Jordan algebras in the booksTaste of Jordan algebras of McCrimmon [14] , Analysis on symmetric cones of Faraut and Korányi [15], and in the book Structure and Representations of Jordan Algebras of Nathan Jacobson [16].
Let $\mathcal{A}$ be a finite dimensional algebra $\mathcal{A}$ over a field $\mathbf{K}$, we denote the product of two elements **a** and **b** of the algebra $\mathcal{A}$ by $\boldsymbol{ab}$. Then $\mathcal{A}$ is a Jordan algebra if for all elements **a** and **b** of $\mathcal{A}$ we have:

$$ab = ba \quad (1)$$

$$a(a^2 b) = a^2 (ab) \quad (2)$$

We can reformulate these properties (1) and (2) using for any **a** and **b** the operators of left multiplication and of right multiplication, L_a and R_b defined by $L_a(x) = ax, x \in A$ and by $R_b(y) = yb, \forall y \in A$. The property (1) is equivalent to say that $L_a(b) = R_a(b)$. And, the property (2) is equivalent to say that $L_a(L_{a^2}) = L_{a^2}(L_a), \forall a \in A$.

If $\mathcal{A}$ is an associative algebra over a field of characteristic distinct from $\frac{1}{2}$, thenif we introduce in $\mathcal{A}$ the Jordan product $a \bullet b = \frac{1}{2}(ab + ba)$ then $\mathcal{A}$ when equipped with this product becomes a Jordan algebra, that is usually denoted by A^{+} .

A Jordan algebra is special if is isomorphic to a subalgebra of some A^{+} Jordan algebra. Otherwise the Jordan algebra is exceptional.

If **A** is an algebra with an involution $*$ then we denote the set of the elements of **A** such that $x* = x$ by Hermitian elements of **A**. And, if **A** is an associative algebra then the set of the Hermitian elements of **A** is a Jordan subalgebra of A^{+}.

For instance,the set of the real matrices of order n, that we denote by $M_n(\mathbb{R})$, is a Jordan algebra with the Jordan product. Indeed, let **x** and **y** be elements of $M_n(\mathbb{R})$. Then we have that:

$$\mathbf{x} \bullet \mathbf{y} = \frac{\mathbf{xy} + \mathbf{yx}}{2} = \frac{\mathbf{yx} + \mathbf{xy}}{2} = \mathbf{y} \bullet \mathbf{x}, \tag{3}$$

and that

$$\begin{aligned} \mathbf{x} \bullet (\mathbf{x}^2 \bullet \mathbf{y}) &= \frac{\mathbf{x}\left(\frac{\mathbf{x}^2\mathbf{y} + \mathbf{yx}^2}{2}\right) + \left(\frac{\mathbf{x}^2\mathbf{y} + \mathbf{yx}^2}{2}\right)\mathbf{x}}{2} \\ &= \frac{\mathbf{x}^2\left(\frac{\mathbf{xy} + \mathbf{yx}}{2}\right) + \left(\frac{\mathbf{xy} + \mathbf{yx}}{2}\right)\mathbf{x}^2}{2} \\ &= \mathbf{x}^2 \bullet (\mathbf{x} \bullet \mathbf{y}). \end{aligned} \tag{4}$$

So we conclude by (3) and (4) that $A = M_n(\mathbb{R})$ is a Jordan algebra with the Jordan product. If we consider the operation transposition that is an involution over the set V of the Hermitian elements of **A**, then V is a Jordan algebra, this is the set of the real symmetric matrices of order n of **A** that we denote by ***Sym(n, R)*** is a Jordan algebra when equipped the Jordan product $\bullet$.

From now on, we only consider Jordan algebras of finite dimension and with element that we will always denote it by **e** and from now on when we say let A be a Jordan algebra we suppose that A is a finite dimensional real Jordan algebra with unit **e**.

An algebra over the field of the reals with unit **e** is power associative if the algebra spanned by **x**is associative. We can say that a Jordan algebra is power associative for a detailed prove one must read proposition II.1.2,page 27 of the book Analysis Symmetric Cones of Faraut and Korányi.

Let A be a n-dimensional Jordan algebra . The rank of **x**in A is the least natural number k such that $\{\mathbf{e}, x, \ldots, x^k\}$ is linearly dependent and we write $\mathrm{rank}(x) = k$. Since $\mathrm{rank}(x) \le n$ the rank of A is defined as being the natural number $\mathrm{rank}(A) = \max\{\mathrm{rank}(x) : x \in A\}$. An element **x** in A is regular if $r = \mathrm{rank}(x) = \mathrm{rank}(A)$. Let x be a regular element of A and $r = ran(x)$.Then, there exist real scalars $a_1(x), a_2(x), \ldots, a_{r-1}(x)$ and $a_r(x)$ such that:

$$\mathbf{x}^{\mathbf{r}} - \mathbf{a}_1(\mathbf{x})\mathbf{x}^{\mathbf{r-1}} + \cdots + (\mathbf{-1})^{\mathbf{r}}\mathbf{a}_{\mathbf{r}}(\mathbf{x})\mathbf{e} = \mathbf{0}. \tag{5}$$

Where **0** is thenull vector of A . Taking in account (5) we conclude that the polynomial **p(x,-)** defined in the equality (6).

$$p(x, \lambda) = \lambda^r - a_1(x)\lambda^{r-1} + \cdots + (-1)^r a_r(x). \tag{6}$$

is the minimal polynomial of **x**. When **x** is not regular the minimal polynomial of **x** has a degree less that **r**and divides the polynomial**p(x,-)**. The roots of the polynomial **p(x,-)** are the eigenvalues of **x**.

Although the characteristic polynomial **p(x,-)** is defined for a regular element of A, we can extend the definition of characteristic polynomial to all elements of A by continuity since each coefficient a_i is a homogeneous polynomial of degree **i**on the coordinates of **x** in a fixed basis of A and the set of regular elements of A is dense in A.

Now, we will explain the reason to call the polynomial **p(x,.)**the characteristic polynomial of **x.**Let**x**be a regular element of the Jordan algebra A and $\mathbf{L}_{0x}$ represents the restriction of the operation of the linear operation L_x to the space $\mathbb{R}[x]$. Now since $rank(x) = r$ then a basis of $\mathbb{R}[x]$ is

$B = \{e, x, x^2, \ldots, x^{r-1}\}$ is a basis of $\mathbb{R}[x]$. Let, express the images of the elements of the basis B by the linear application L_{0x} on the basis B. Hence, we present the following calculations:

$$\begin{aligned}
&L_{0x}(e) = x = 0e + x + \cdots + 0x^{r-1},\\
&L_{0x}(x) = x^2 = 0e + 0x + x^2 + \cdots + 0x^{r-1},\\
&L_{0x}(x^2) = x^3 = 0e + 0x + 0x^2 + x^3 + \cdots + 0x^{r-1},\\
&\vdots\\
&L_{0x}(x^{r-2}) = x^{r-1} = 0e + 0x + 0x^2 + 0x^3 + \cdots + 0x^{r-2} + x^{r-1},\\
&L_{0x}(x^{r-1}) = x^r = -(-1)^r a_r(x)e - (-1)^{r-1} a_{r-1}(x)x - \cdots + a_1(x)x^{r-1}.
\end{aligned}$$

Therefore the matrix $M_{L_{0x}}$ of the operator L_{0x} is

$$M_{L_{0x}} = \begin{bmatrix} 0 & 0 & \cdots & -(-1)^r a_r(x) \\ 1 & 0 & \cdots & -(-1)^{r-1} a_{r-1}(x) \\ 0 & 1 & \cdots & -(-1)^{r-2} a_{r-2}(x) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -(-a_1(x)) \end{bmatrix}$$

Now the characteristic polynomial of the matrix $M_{L_{0x}}$ is the polynomial p such that $p(\lambda) = \left|\lambda I_r - M_{L_{0x}}\right| = \lambda^r - a_1(x)\lambda^{r-1} + \cdots + (-1)^r a_r(x)$. This is, we had obtained that $p(\lambda) = p(x,\lambda)$. So it is natural to designated the polynomial $p(x,\lambda)$ by characteristic polynomial of **x**when **x** is a regular element of A.

We remember again that the roots of the characteristic polynomial $p(x,\lambda)$ are called the eigenvalues of **x** and since trace of the matrix $M_{L_{0x}}$ is $a_1(x)$ and the determinant of $M_{L_{0x}}$ is $a_r(x)$ then we call $a_1(x)$ the trace of **x** and $a_r(x)$ the determinant of **x**.

A real Euclidean Jordan algebra $\mathcal{A}$ is a Jordan algebra with an inner product $<-,->$ such that $<x \bullet y, z> = <y, x \bullet z>$ for all **x**, y and **z** in $\mathcal{A}$. Herein, we must say that using the linear operator of multiplication by left L_x then we conclude that A is an Euclidean Jordan with an inner product $<-,->$ if $< L_x(y), z > = < y, L_x(z) >, \forall x, y, z \in A$.

The real vector space of real symmetric matrices of order n, $\mathrm{Sym}(n,\mathbb{R})$, is a real Euclidean Jordan algebra when $Sym(n,\mathbb{R})$ is equipped with the Jordan product $\bullet$ defined as follows: $x \bullet y = (xy + yx)/2$ where xy and yx denotes the usual product of the matrices **x** and **y** respectively**,** and the inner product $<-,->$ defined by $<x, y> = \mathrm{tr}(x \bullet y)$, where **tr** denotes the usual trace of matrices. The unit element of this Euclidean Jordan algebra is the identity matrix of order n, I_n. Indeed, for **x,y** and **z** in $Sym(n,\mathbb{R})$ we have:

$$\begin{aligned}
< L_x(y), z) &= \mathrm{tr}(L_x(y)z)\\
&= \mathrm{tr}((x \bullet y)z)\\
&= \mathrm{tr}\left(\frac{xy + yx}{2} z\right)\\
&= \mathrm{tr}\left(\frac{(xy)z + (yx)z}{2}\right)\\
&= \mathrm{tr}\left(\frac{xyz + yxz}{2}\right)\\
&= \mathrm{tr}\left(\frac{xyz}{2}\right) + \mathrm{tr}\left(\frac{yxz}{2}\right)\\
&= \mathrm{tr}\left(\frac{yxz}{2}\right) + \mathrm{tr}\left(\frac{yzx}{2}\right)\\
&= \mathrm{tr}\left(\frac{yxz}{2} + \frac{yzx}{2}\right)\\
&= \mathrm{tr}\left(y\left(\frac{xz + zx}{2}\right)\right)\\
&= \mathrm{tr}(y(L_x(z)))\\
&= < y, L_x(z) >.
\end{aligned}$$

Let $\mathcal{A}$ be a real Euclidean Jordan algebra with unit element $\mathbf{e}$. An element $\mathbf{c}$ in $\mathcal{A}$ is an idempotent if $c^2 = c$. Two idempotents $\mathbf{c}$ and $\mathbf{d}$ are orthogonal if $c \bullet d = 0$. Let $\mathbf{k}$ be a natural number. The set $\{g_1, g_2, \cdots, g_{k-1}, g_k\}$ is a complete system of orthogonal idempotents if the following three conditions hold: (*i*) $g^2{}_i = g_i$, for $i = 1, \cdots, k$, (*ii*) $g_i \quad g_j = 0$ if $i \neq j$, and (*iii*) $\sum^k{}_{i=1} g_i = e$. An idempotent $\mathbf{c}$of $\mathcal{A}$ is primitive if it is a nonzero idempotent of $\mathcal{A}$ and if it can not be written as a sum of two non-null orthogonal idempotents. We say that $\{g_1, g_2, \cdots, g_{k-1}, g_k\}$ is a Jordan frame if it is a complete system of orthogonal idempotents such that each idempotent is primitive.

Proposition1. ([15], pp.43).

Let A be a real Euclidean Jordan algebra and $\mathbf{x}$ in A. Then there exist unique real number $\lambda_1, \lambda_2, \cdots \lambda_{k-1}$ and λ_k all distinct, and a unique complete system of orthogonal idempotents $\{g_1, g_2, \cdots, g_{k-1}, g_k\}$ such that

$$x = \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_k g_k \tag{7}$$

The numbers λ_j s for $i = 1, \cdots, k$ of (7) are the eigenvalues of $\mathbf{x}$ and the decomposition (7) is called the first spectral decomposition of $\mathbf{x}$.

Example 1.

Let consider the Euclidean Jordan algebra $\mathbf{A} = Sym(n, \mathbb{R}^n)$ and $B \in \mathbf{A}$ a matrix with the distinct non null eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{k-1}$ and λ_k. Then the set $S = \{P_1, P_2, \cdots, P_{k-1}, P_k\}$ with $P_i = \prod_{1 \leq j \leq k, j \neq i} (B - \lambda_j I_n) / (\lambda_i - \lambda_j)$ is a complete system of orthogonal idemptents of $\mathbf{A}$ and the first spectral decomposition of $\mathbf{B}$ is

$$B = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_k P_k.$$

Proposition2 ([15], pp.44)

Let A be a real Euclidean Jordan algebra such that $r = rank(A)$ and $x \in A$. Then there exists a Jordan frame $\{g_1, g_2, \cdots, g_r\}$ and real numbers $\lambda_1, \lambda_2, \cdots, \lambda_{r-1}$ and λ_r such that

$$x = \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_r g_r. \tag{8}$$

The equality (8) is called the second spectral decomposition of $\mathbf{x}$.

Example 2

Let $A = Sym(n, \mathbb{R})$ be the Euclidean Jordan algebra with the Jordan product $x \bullet y = \dfrac{xy + yx}{2}$ and with the inner product $< x, y >= tr(x \bullet y)$ and let $\mathbf{C}$ be a real symmetric of A., and let $S = \{v_1, v_2, \cdots, v_n\}$ be an orthonormal basis of eigenvectors of $\mathbf{C}$ such that $Cv_i = \lambda_i v_i, \forall i = 1, \ldots, n$.

We suppose that we are adopting a column notation, this is we consider that

$$\mathbf{v_i} = \begin{bmatrix} \mathbf{v_{1i}} \\ \mathbf{v_{2i}} \\ \vdots \\ \mathbf{v_{n_i i}} \end{bmatrix}, \forall \mathbf{i} = \mathbf{1}, \cdots, \mathbf{n}$$

Let consider $g_i = v_i v^T{}_i, i = 1, \cdots, n$, then $B = \{g_1, g_2, \cdots, g_n\}$ is a Jordan frame. I

Indeed, let $\mathbf{i}$be a natural number less or equal$\mathbf{n}$, we have:

$$g^2{}_i = g_i \quad g_i = g_i g_i = v_i v^T{}_i v_i v^T{}_i = v_i \left(v^T{}_i v_i\right) v^T{}_i = v_i (1) v^T{}_i = g_i,$$

so each g_i is an idempotent element of the Euclidean Jordan algebra A. Let $\mathbf{i}$ and $\mathbf{j}$be two distinct natural numbers less or equal $\mathbf{n}$thenwe have

$$g_i \bullet g_j = \left(v_i v^T{}_i\right) \bullet \left(v_j v^T{}_j\right) = \frac{\left(v_i v^T{}_i\right)\left(v_j v^T{}_j\right) + \left(v_j v^T{}_j\right)\left(v_i v^T{}_i\right)}{2} = \frac{\left(v_i (v^T{}_i v_j) v^T{}_j\right) + \left((v_j (v^T{}_j v_i) v^T{}_i\right)}{2}$$

$$= \frac{\left(v_i 0 v^T{}_j\right) + \left((v_j 0 v^T{}_i\right)}{2} = O_n,$$

where O_n is null real matrix of order n. Then the idempotents g_i and g_j are orthogonal for $i \neq j$.

Now, since **S** is an orthonormal basis of $\mathbb{R}^n$ then we have

$$v_1 v^T{}_1 + v_2 v^T{}_2 + \cdots + v_n v^T{}_n = I_n$$

this is, we have

$$g_1 + g_2 + \cdots + g_n = \mathbf{I}_n$$

Hence, we have proved that B is a Jordan frame of the Euclidean Jordan algebra A since $rank(A) = n.$
Finally, we will obtain the second spectral decomposition of **C**.

$$\begin{aligned} C &= CI_n \\ &= C(g_1 + g_2 + \cdots + g_n) \\ &= C\left(v_1 v^T{}_1 + v_2 v^T{}_2 + \cdots + v_n v^T{}_n\right) \\ &= (Cv_1)v^T{}_1 + (Cv_2)v^T{}_2 + \frac{1}{2}\cdots + (Cv_n)v^T{}_n \\ &= (\lambda_1 v_1)v^T{}_1 + (\lambda_2 v_2)v^T{}_2 + \cdots + (\lambda_n v_n)v^T{}_n \\ &= \lambda_1 v_1 v^T{}_1 + \lambda_2 v_2 v^T{}_2 + \cdots + \lambda_n v_n v^T{}_n \\ &= \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_n g_n. \end{aligned}$$

Then, the second spectral decomposition of **C** is:

$$C = \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_n g_n.$$

III. RESULTS ON STRONGLY REGULARSGRAPHS

R. C. Bose in 1963 in [17] introduced the strongly regular graphs and partial balanced designs.
A good survey on algebraic graph theory and particularly on strongly regular graph is the book of C. Godsil and G. Royle , Algebraic graph theory of C. Godsil and G. Royle [18].
A non null and non complete graph **G** is a $(n,k;\lambda,\mu)$ strongly regular graph if **G** is **a k-**regular graph,**k>1,**of order **n**, **n>3,** and any two adjacent vertices have λ common neighbors and any two nonadjacent vertices have μ common neighbors.
Let**G**be a $(n,k;\lambda,\mu)$ strongly regular graphthen we have the admissibility condition $k(k-\lambda-1) = \mu(n-k-1).$ The adjacency matrix **A** of **G**satisfies the equality (9).

$$A^2 = kI_n + \lambda A + \mu(J_n - A - I_n). \tag{9}$$

The eigenvalues of G are k , θ and τ , where θ and τ are given by $\theta = (\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)})/2$ and $\tau = (\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)})/2$.The complement of the $(n,k;\lambda,\mu)$ strongly regular graph Gis an $(n, n-k-1; n-2k+\mu-2, n-2k+\lambda)$ strongly regular grap. The multiplicities of the eigenvalues θ and τ of **G**are m_θ and m_τ and their expressions are given in (10) and (11).

$$m_\theta = \frac{/\,z\,/\,n + \tau - k}{\theta - \tau} \tag{10}$$

$$m_\tau = \frac{\theta n + k - \theta}{\theta - \tau} \tag{11}$$

The Krein admissibility conditions established by Jr. L. L. Scott in [19] of a $(n,k;\lambda,\mu)$ strongly regular graph , are presented in the inequalities (12) and (13).

$$(k+\theta+2\theta\tau)(\theta+1) \le (k+\theta)(\tau+1)^2, \tag{12}$$

$$(k+\tau+2\theta\tau)(\tau+1) \le (k+\tau)(\theta+1)^2. \tag{13}$$

Now, we presentthe admissibility conditions on the parameters of a $(n,k;\lambda,\mu)$ strongly regular graphintroduced by Delsarte, Goethals and Seidel[20], in inequalities (14) and (15).

$$n \le \frac{m_\theta\left(m_\theta+3\right)}{2}, \tag{14}$$

$$n \le \frac{m_\tau\left(m_\tau+3\right)}{2}. \tag{15}$$

A $(n,k;\lambda,\mu)$ strongly regular graph **G** is primitive if and only if **G** is connect and its complement $\bar{G}$ is also connected. A strongly regular graph that is not primitive is called imprimitive.
A $(n,k;\lambda,\mu)$ strongly regular graph **G** is a imprimitive strongly regular graph if and only if $\mu = 0$ or $\mu = k.$
From now, when we say let **G** be a $(n,k;\lambda,\mu)$ strongly regular graph we suppose that **G** is a primitive strongly regular graph.

IV. ADMISSIBILITY CONDITIONS ON THE PARAMETERSOF A STRONGLY REGULAR GRAPH

In this section, we will establish inequalities on the parameters of a strongly regular graph and on his spectra, in an algebraic way, in the environment of Euclidean Jordan algebras .

Let G be a $(n,k;\lambda,\mu)$ strongly regular graph with $0<\mu<k<n-1$ and $k<\frac{n}{2}$ and let A be its adjacency matrix with the distinct eigenvalues, namely k , θ and τ , and let $\mathcal{A}$ be the 3 dimensional real Euclidean subalgebra of the Euclidean Jordan algebra $Sym(n,R)$ spanned by I_n and the natural powers of A. The Euclidean Jordan algebra $\mathcal{A}$ is such that $rank(\mathcal{A}) = 3$

Let $S=\{E_1,E_2,E_3\}$ be the complete system of orthogonal idempotents of A associated to $\mathcal{A}$, where $E_1 = 1/nI_n + 1/nA + 1/n(J_n - A - I_n)$, $E_2 = (|\tau|n+\tau-k)/(n(\theta-\tau))I_n + (n+\tau-k)/(n(\theta-\tau))$ $A + (\tau-k)/(n(\theta-\tau))(J_n - A - I_n)$ and $E_3 = (\theta n + k - \theta)/(n(\theta-\tau))I_n + (-n+k-\theta)/(n(\theta-\tau))A$ $+(k-\theta)/(n(\theta-\tau))(J_n - A - I_n)$.
Leti and **j** be natural numbers such that $1 \le i,j \le 3$ and $i \ne j.$ So, since the idempotents E_i and E_j are orthogonal relatively to the Jordan product of matrices, then they are orthogonal relatively to the inner product $<x,y> = tr(x\ \ y), \forall x,y \in \mathcal{A}$. Therefore $S=\{E_1,E_2,E_3\}$ is a basis of $\mathcal{A}$.
Let consider some notation for defining the Hadamard product of two matrices. For two matrices of order n, E and F of $M_n(\mathbb{R})$ .We define $E \circ F$ in the following way : if $E=[e_{ij}]$ and $F=[f_{ij}]$, then $E \circ F = [e_{ij}f_{ij}]$ for all $i,j \in \{1,\ldots,n\}.$ For any natural number l and for any matrix $H \in M_n(\mathbb{R})$ we define $H^{\circ 0} = J_n, H^{\circ 1} = H$, and $H^{\circ l} = H \circ H^{\circ(l-1)}, l \ge 2$ see[21].
Now, we will analyze the spectra of an Hadamard series associated to the adjacency matrix A of G. Let consider the Hadamard series $S_X = \sum_{j=1}^{+\infty}\frac{1}{j}\left(E_3^{\circ 2}\right)^{\circ j}.$

We have that :

$$S_X = \sum_{j=1}^{+\infty}\frac{1}{j}\left(\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2 I_n + \left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2 A + \left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2 (J_n - A - I_n)\right)^{\circ j}$$

$$= \sum_{j=1}^{+\infty}\frac{1}{j}\left(\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right)^j I_n + \sum_{j=1}^{+\infty}\frac{1}{j}\left(\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2\right)^j A + \sum_{j=1}^{+\infty}\frac{1}{j}\left(\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)^j (J_n - A - I_n).$$

$$= -\ln\left(1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right) I_n - \ln\left(1-\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2\right) A - \ln\left(1-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)(J_n - A - I_n).$$

Now, we consider the following notations:

$$S_x = q_{1x}E_1 + q_{2x}E_2 + q_{3x}E_3 \qquad (16)$$

$$S_{nx} = q_{1nx}E_1 + q_{2nx}E_2 + q_{3nx}E_3 \qquad (17)$$

where $S_{nx} = \sum_{j=1}^{n}\left(E_3{}^2\right)^{\circ j}$.

The eigenvalues q_{inx} for $i = 1, \cdots, 3$ of the second spectral decomposition (17) of S_{nx} are positive since **S**isa Jordan frameof $\boldsymbol{A}$ and $\boldsymbol{A}$ is closed for the Hadamard operation of two elements of $\boldsymbol{A}$ and since for any two real matrices **E** and **F** of order n we have $\lambda_{min}(E)\lambda_{min}(F) \le \lambda_{min}(E \circ F)$ and therefore the eigenvalues $q_{ix}, \forall i = 1, \ldots, 3$ of the second spectral (16) of S_x, since we have $q_{ix} = \lim_{n \uparrow \infty} q_{inx}, \forall\ i = 1, \ldots, 3$.

Now we consider the element $E_3 \circ S_x$ and the notation $E_{33x} = (E_3)^2 \circ S_x = q_{13x}E_1 + q_{23x}E_2 + q_{33x}E_3$. Since S_x and $(E_3)^2$ have positive eigenvalues then by the properties enunciated before we conclude that the q_{i3x} s are positive. The expressions of these eigenvalues are present in the (18),(19) and (20).

$$q_{13x} = -\frac{\theta n+k-\theta}{n(\theta-\tau)}\ln\left(1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right) - \frac{-n+k-\theta}{n(\theta-\tau)}\ln\left(1-\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2\right)k - \frac{k-\theta}{n(\theta-\tau)}\ln\left(1-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)(n-k-1) \qquad (18)$$

$$q_{23x} = -\frac{\theta n+k-\theta}{n(\theta-\tau)}\ln\left(1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right) - \frac{-n+k-\theta}{n(\theta-\tau)}\ln\left(1-\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2\right)\tau - \frac{k-\theta}{n(\theta-\tau)}\ln\left(1-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)(-\tau-1), \qquad (19)$$

$$q_{33x} = -\frac{\theta n+k-\theta}{n(\theta-\tau)}\ln\left(1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right) - \frac{-n+k-\theta}{n(\theta-\tau)}\ln\left(1-\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2\right)\theta - \frac{k-\theta}{n(\theta-\tau)}\ln\left(1-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)(-\theta-1). \qquad (20)$$

Now, by the analysis of the parameters q_{13x} we establish the inequality (21) of Proposition 3.

Proposition 3.

Let G be a $(n,k;\lambda,\mu)$-strongly regular graph with $0 < \mu < k < n-1$ and with the distinct eigenvalues k, θ and τ. If $k < n/2$ then

$$\left(\frac{(n(\theta-\tau))^2-(k-\theta)^2}{(n(\theta-\tau))^2-(\theta n+k-\theta)^2}\right)^{\frac{\theta n+k-\theta}{n-k+\theta}} > \left(\frac{(n(\theta-\tau))^2-(k-\theta)^2}{(n(\theta-\tau))^2-(n-k+\theta)^2}\right)^k. \qquad (21)$$

Proof

Since $q_{13x} \ge 0$ then we have:

$$-\frac{\theta n+k-\theta}{n(\theta-\tau)}\ln\left(1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^2\right) - \frac{-n+k-\theta}{n(\theta-\tau)}\ln\left(1-\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2\right)k - \frac{k-\theta}{n(\theta-\tau)}\ln\left(1-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right)(n-k-1) > 0. \qquad (22)$$

Now since $\frac{\theta n+k-\theta}{n(\theta-\tau)} + \frac{-n+k-\theta}{n(\theta-\tau)}k + \frac{k-\theta}{n(\theta-\tau)}(n-k-1) = 0$ then from (22) we deduce (23).

$$\frac{\theta n+k-\theta}{n(\theta-z)}\left(ln\left(1-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^{2}\right)-ln\left(1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^{2}\right)\right)-\frac{n-k+\theta}{n(\theta-\tau)}\left(ln\left(1-\left(\frac{k-\theta}{n(\theta-\tau)}\right)^{2}\right)-ln\left(1-\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^{2}\right)\right)k\geq 0. \quad (23)$$

Therefore using the properties of logarithms in the equality (23) we obtain (24).

$$\frac{\theta n+k-\theta}{n(\theta-z)}ln\left(\frac{(n(\theta-\tau))^{2}-(k-\theta)^{2}}{(n(\theta-\tau))^{2}-(\theta n+k-\theta)^{2}}\right)-\frac{n-k+\theta}{n(\theta-\tau)}ln\left(\frac{(n(\theta-\tau))^{2}-(k-\theta)^{2}}{(n(\theta-\tau))^{2}-(n-k+\theta)^{2}}\right)k\geq 0. \quad (24)$$

Then after rewriting (24) we deduce the inequality (25).

$$ln\left(\frac{(n(\theta-\tau))^{2}-(k-\theta)^{2}}{(n(\theta-\tau))^{2}-(\theta n+k-\theta)^{2}}\right)\geq\frac{n-k+\theta}{\theta n+k-\theta}ln\left(\frac{(n(\theta-\tau))^{2}-(k-\theta)^{2}}{(n(\theta-\tau))^{2}-(n-k+\theta)^{2}}\right)k. \quad (25)$$

Therefore, using the properties of the logarithms in inequality (25) we obtain (26).

$$\left(\frac{(n(\theta-\tau))^{2}-(k-\theta)^{2}}{(n(\theta-\tau))^{2}-(\theta n+k-\theta)^{2}}\right)^{\frac{\theta n+k-\theta}{n-k+\theta}}\geq\left(\frac{(n(\theta-\tau))^{2}-(k-\theta)^{2}}{(n(\theta-\tau))^{2}-(n-k+\theta)^{2}}\right)^{k}. \quad (26)$$

V. NUMERICAL RESULTS AND CONCLUSIONS

In this section we present some examples of parameters $(n,k;\lambda,\mu)$ that don't satisfy the inequality (26) of Proposition 3. In table 1 we consider the parameters sets $P_1=(56,22;3,12)$, $P_2=(81,40;13,26)$, $P_3=(300,92;10,36)$ and $P_4=(405,84;3,21).$ We present in the Table 1 the eigenvalues θ and τ , and the value of the parameter $q_{\theta\tau kn}$ defined on the equality (27) obtained from a rewriting of the inequality (26) of Proposition 3.

$$q_{\theta\tau kn}=\left(\frac{(n(\theta-\tau))^{2}-(k-\theta)^{2}}{(n(\theta-\tau))^{2}-(\theta n+k-\theta)^{2}}\right)^{\frac{\theta n+k-\theta}{n-k+\theta}}-\left(\frac{(n(\theta-\tau))^{2}-(k-\theta)^{2}}{(n(\theta-\tau))^{2}-(n-k+\theta)^{2}}\right)^{k}. \quad (27)$$

	$P_1=(56,22;3,12)$	$P_2=(81,40;13,2$	$P_3=(300,92;10,36$	$P_5=(405,84;$
θ	**1**	**1**	**2**	**3**
τ	**-10**	**-14**	**-28**	**-21**
$q_{\theta\tau kn}$	**-0.013999650**	**0.018765036**	**-0.022565595**	**-0.017409481**

Table 1

The parameters sets presented in the Table 1satisfy the requirements of the Proposition 3 and the corresponding results satisfy the fact that the parameter $q_{\theta\tau kn}$ is negative when the Krein conditions (12) and (13) are not satisfied, but for all the parameters setsexperimenteduntil now that satisfied the upper conditions (14) and (15) the parameter $q_{\theta\tau kn}$ had presented positive eigenvalues.

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REFERENCE

[1]. F. Alizadeh and S. H. Schmieta, " Symmetric Cones, Potential Reduction Methods and Word-By Word Extensions" in R. Saigal , L. Vandenberghe and H. Wolkowicz, editors, Handbook of Semidefinite Programming, Theory, Algorithms and Applications, pages, 195-233, Kluwer Academic Publishers, 2000.
[2]. D. M. Cardoso and L. A. Vieira, "On the optimal parameter of a self-concordant barrier over a symmetric cone". European Journal of Operational Research., 169: 1148-1157, 2006.
[3]. L. Faybusovich, "Euclidean Jordan Algebras and Interior-point Algorithms".Positivity, 1,:331-357, 1997.

[4]. M.S. Gowda and J. Tao, "Some Inertia Theorems on Euclidean Jordan Algebras", Linear algebra and applications, 430(8-9):1991-2011, 2009.
[5]. M. S. Gowda and M. M. Moldovan, "More results on Schur complements in Euclidean Jordan algebras", Journal of Global Optimization., 53(1):121-134, 2012.
[6]. M. S. Gowda, R. Sznajder, and J. Tao, " Some P-properties for linear transformations on Euclidean Jordan algebras ", Linear Algebra and its Applications, 320:131-144, 2000.
[7]. D. M. Cardoso, and L. A. Vieira, "Euclidean Jordan Algebras with strongly regular graphs", Journal of Mathematical Sciences, 120: 881-894, 2004.
[8]. V. M. Mano, E. A. Martins and L. A. Vieira, "On generalized binomial series and strongly regular graphs", Proyecciones Journal of Mathematics, 4: 393-408, 2011.
[9]. V. M. Mano, L. A. Vieira, "Admissibility Conditions and Asymptotic Behaviour of Strongly Regular Graph", International Journal of Mathematical Models and Methods in Applied Sciences, 5(6):1027-1033, 2011.
[10]. V. M. Mano and L. A Vieira, "Alternating Schur Series and Necessary Conditions for the Existence of strongly Graphs",International Journal of Mathematical Models and Methods in Applied Sciences Methods, 8:256-261, 2014.
[11]. L. A. Vieira and V. M. Mano, "Generalized Krein parameters of a strongly regular graph", Applied Mathematics, 6: 37-45, 2015.
[12]. L. A. Vieira, "Asymptotic Properties of the Spectra of a Strongly Regular Graph" , in Innovation, Engineering and Entrepreneurship, Lecture notes in Electrical Engineering, book series 505 edited by J. Machado, F. Soares and G. Veiga. Springer-Verlag:800-804, 2018.
[13]. J. D. Malley, Statistical Applications of Jordan Algebras, Springer-Verlag, New York, 2000.
[14]. K. McCrimmon, A taste on Jordan Algebras, Springer, New York, 2000.
[15]. J. Faraut and A. Korányi, Analysis on Symmetric Cones, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1994.
[16]. N. Jacobson, The Structure and Representations of Jordan Algebras, American Mathematical Society, Rhode Island, 1968.
[17]. B. C. Bose, " Strongly regular graphs, partial geometries and partially balanced designs", Pacific J. Math., 13: 389-419, 1963.
[18]. C. Godsil and G. Royle, Algebraic Graph Theory, Chapman&Hall, New York, 1993.
[19]. Jr. L. L. Scott,"A condition on Higman's parameters" ,Notices Amerc. Math. Soc., 20: A-97, 1973.
[20]. P. Delsart, J. M. Goethals and J.J. Seidel, "Bounds for systems of lines and Jacobi polynomials", Philips Res Repts, 30:91-95, 1975.
[21]. R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.

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