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Theoretical Matrix Study of Rigid Body Absolute Motion

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--ABSTRACT--
The absolute motion of an arbitrary asymmetric rigid body is studied. This motion is determined after its relative motion has been obtained. The most important peculiarity in this dynamical rigid body model is that the selected pole does not coincide with its mass center. Seven new kinematic characteristics have been defined. The first ones are the following vectors: real absolute, transmisive and relative generalized velocities. The second ones are the vectors-real absolute, transmisive, relative and Coriolisian generalized accelerations. Two new theorems are formulated. The first one is for summation the vectors of real generalized velocities. The second one is for summation the vectors of real generalized accelerations. The system of differential equations describing the rigid body relative motion in matrix form is determined. The algorithm for obtaining the rigid body absolute motion is described.

***KEYWORDS** – matrix study, rigid body, absolute motion, relative motion.*

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I. INTRODUCTION

This article represents an extension of the ideas described in the work [1]. If the investigations on the rigid body general motions are traced in historical plan, it will be ascertained that at the beginning of Mechanics development, the rigid bodies have been assumed as symmetrical, [2, 3]. This statement has its own explanations. Then there were no computational tools and the solutions were mostly analytical, [4, 5, 6]. However, with development of mathematics and numerical methods, the researches were gradually exploring the three-dimensional rigid body motions of not only symmetrical but also asymmetric bodies.

This complication of the dynamic model coincides with the development of matrices and matrix calculations, [7]. In these cases, the initial point of body-related coordinate systems, called shortly pole, is chosen to coincide with the rigid body mass center. This leads to significant advantages and simplifies mainly in the type of differential equations describing three-dimensional rigid body motions, [8, 9, 10, 11].

In the present work, the most complex dynamic model is used – an asymmetric rigid body with a pole that does not coincide with its mass center. Description of this most complex dynamic model requires the using of new additional kinematic and dynamic characteristics as well as new theorems in rigid body Kinematics and Dynamics, [1, 12, 13].

The study of the relative and absolute motion of a material point is described in many Mechanics books, for example [14, 15]. Theoretical matrix study of relative and absolute motion of an asymmetric rigid body with a pole that does not coincide with its mass center is a very important problem for the engineering practice. That is why, this actual task is precisely studied in this work.

II. STATEMENT OF THE PROBLEM

The general motion of a free asymmetric rigid body marked with the letter L is studied, (Fig.1).

The body A is called absolutely body. It is assumed absolutely immovable.

The coordinate system $N\xi\eta\zeta$ defined with unit vectors $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ is fixedly connected to this body A. It is called absolutely coordinate system. All vectors and matrices counted to the coordinate system $N\xi\eta\zeta$ or any other coordinate system which is moving translational to the coordinate system $N\xi\eta\zeta$ are denoted by a lower index A.

The body B is called transmission body. It is making a general absolutely motion towards body A.

An arbitrary point O_1 from the body B is chosen. This point is called shortly pole. Its position in the space of coordinate system $N\xi\eta\zeta$ is defined by absolutely radius-vector $\boldsymbol{\rho}_{O_1}$.

Two coordinate systems are put in the pole O_1.

The first coordinate system $O_1\xi_1\eta_1\zeta_1$ is moving translational towards the coordinate system $N\xi\eta\zeta$.

The second coordinate system $O_1 x_1 y_1 z_1$ is fixedly connected to the body B and it is defined by unit vectors $\mathbf{e}_{x_1}$, $\mathbf{e}_{y_1}$ and $\mathbf{e}_{z_1}$. The orientation of the axes of the coordinate system $O_1 x_1 y_1 z_1$ towards the coordinate system $O_1 \xi_1 \eta_1 \zeta_1$ is set by three Cardan angles ψ_e, θ_e and φ_e. All vectors and matrices counted to the coordinate system $O_1 x_1 y_1 z_1$ are denoted by a lower index B. All vectors and matrices counted to the coordinate system $O_1 \xi_1 \eta_1 \zeta_1$ are denoted by a lower index A.

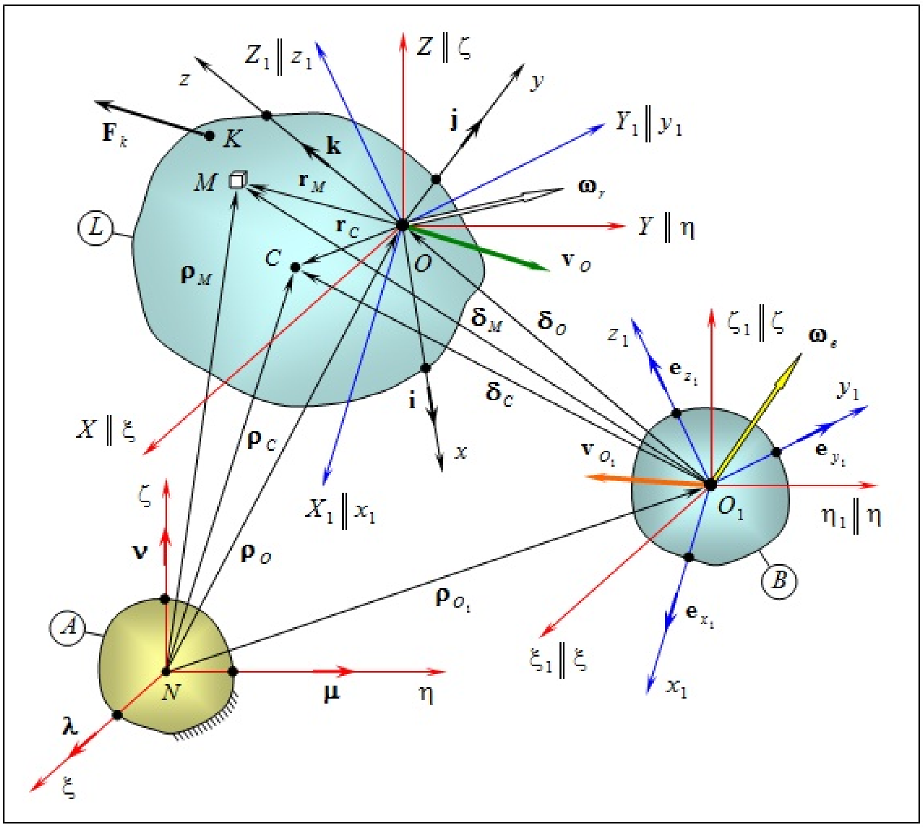

Fig. 1 Dynamics of rigid body absolute motion.

Between the coordinate systems $N\xi\eta\zeta$ and $O_1 x_1 y_1 z_1$, or between the coordinate systems $O_1 \xi_1 \eta_1 \zeta_1$ and $O_1 x_1 y_1 z_1$, the following transition matrices are introduced:

$$\mathbf{U}_{A,B} = \mathbf{U}_{B,A}^{T}, \qquad \mathbf{U}_{B,A} = \mathbf{U}_{A,B}^{T}. \tag{1}$$

The main kinematic characteristics for body B are the following.

Absolutely velocity of the pole O_1:

$$\mathbf{v}_{O_1,A}^{(a)} = \left\langle \; v_{O_1,\zeta_1} \quad v_{O_1,\eta_1} \quad v_{O_1,\zeta_1} \; \right\rangle^{T}. \tag{2}$$

Vector-transmission angle velocity:

$$\boldsymbol{\omega}_{A}^{(e)} = \left\langle \; \omega_{\xi_1}^{(e)} \quad \omega_{\eta_1}^{(e)} \quad \omega_{\zeta_1}^{(e)} \; \right\rangle^{T}. \tag{3}$$

Matrix- transmission angle velocity:

$$\boldsymbol{\Omega}_{A}^{(e)} = \begin{bmatrix} 0 & -\omega_{\zeta_1}^{(e)} & \omega_{\eta_1}^{(e)} \\ \omega_{\zeta_1}^{(e)} & 0 & -\omega_{\xi_1}^{(e)} \\ -\omega_{\eta_1}^{(e)} & \omega_{\xi_1}^{(e)} & 0 \end{bmatrix}. \tag{4}$$

Body L is called relative body. It is making absolutely motion towards body A and at the same time it is making relative motion towards body B. For a pole of body L is chosen an arbitrary point O, which do not coincide with rigid body mass center C. The pole O is defined by absolutely radius-vector $\boldsymbol{\rho}_O$ and by relative radius-vector $\boldsymbol{\delta}_O$. The mass center C is defined by absolutely radius-vector $\boldsymbol{\rho}_C$, by the relative radius-vector $\boldsymbol{\delta}_C$, and the radius-vector $\mathbf{r}_C$.

Three coordinate systems are put in the pole O.

The first coordinate system $O\,X\,Y\,Z$ is moving translational to the coordinate system $N\,\xi\,\eta\,\zeta$. The second coordinate system $O\,X_1\,Y_1\,Z_1$ is moving translational to the coordinate system $O_1\,x_1\,y_1\,z_1$. The third coordinate system $O\,x\,y\,z$ is fixedly connected to this body L and it is defined by unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. The orientation of the axes of the coordinate system $O\,x\,y\,z$ towards the coordinate system $O_1\,x_1\,y_1\,z_1$ is set by three Cardan angles ψ_r, θ_r and φ_r. The orientation of the axes of the coordinate system $O\,x\,y\,z$ towards the coordinate system $O\,X\,Y\,Z$ is set by three Cardan angles ψ_a, θ_a and φ_a. All vectors and matrices counted to the coordinate system $O\,x\,y\,z$ are denoted by a lower index L.

Between the coordinate systems $O\,x\,y\,z$ and $O\,X_1\,Y_1\,Z_1$, or between the coordinate systems $O\,x\,y\,z$ and $O_1\,x_1\,y_1\,z_1$, the following transition matrices are introduced:

$$\mathbf{U}_{B,L} = \mathbf{U}_{L,B}^{T} \;, \qquad \mathbf{U}_{L,B} = \mathbf{U}_{B,L}^{T} \;. \tag{5}$$

Between the coordinate systems $O\,x\,y\,z$ and $O\,X\,Y\,Z$, or between the coordinate systems $O\,x\,y\,z$ and $N\,\xi\,\eta\,\zeta$, the following transition matrices are introduced:

$$\mathbf{U}_{A,L} = \mathbf{U}_{A,B}.\mathbf{U}_{B,L} = \mathbf{U}_{L,A}^{T} \;, \qquad \mathbf{U}_{L,A} = \mathbf{U}_{L,B}.\mathbf{U}_{B,A} = \mathbf{U}_{A,L}^{T} \;. \tag{6}$$

The main kinematic characteristics for body B are the following.

Relative velocity of the pole O:

$$\mathbf{v}_{O,A}^{(r)} = \left\langle\; \dot{\delta}_{O,X} \quad \dot{\delta}_{O,Y} \quad \dot{\delta}_{O,Z} \;\right\rangle^{T} \;. \tag{7}$$

Vector-relative angle velocity:

$$\boldsymbol{\omega}_{A}^{(r)} = \left\langle\; \omega_{X}^{(r)} \quad \omega_{Y}^{(r)} \quad \omega_{Z}^{(r)} \;\right\rangle^{T} \;. \tag{8}$$

Matrix- relative angle velocity:

$$\boldsymbol{\Omega}_{A}^{(r)} = \begin{bmatrix} 0 & -\,\omega_{Z}^{(r)} & \omega_{Y}^{(r)} \\ \omega_{Z}^{(r)} & 0 & -\,\omega_{X}^{(r)} \\ -\,\omega_{Y}^{(r)} & \omega_{X}^{(r)} & 0 \end{bmatrix} . \tag{9}$$

Using the kinematic characteristics defined above, the velocity of the pole O is determined by the formula:

$$\mathbf{v}_{O,A}^{(a)} = \mathbf{v}_{O_1,A}^{(a)} + \boldsymbol{\Delta}_{O,A}^{T}.\boldsymbol{\omega}_{A}^{(e)} + \mathbf{v}_{O,A}^{(r)} \;. \tag{10}$$

The following new matrix is introduced in expression (10), namely:

$$\boldsymbol{\Delta}_{O,A}^{T} = \begin{bmatrix} 0 & \delta_{O,\zeta} & -\,\delta_{O,\eta} \\ -\,\delta_{O,\zeta} & 0 & \delta_{O,\xi} \\ \delta_{O,\eta} & -\,\delta_{O,\xi} & 0 \end{bmatrix} . \tag{11}$$

III. NEW GENERALIZED KINEMATIC CHARACTERISTICS

The following **new generalized kinematic characteristics** are defined.

Vector-real absolute generalized velocity at the pole O:

$$\mathbf{u}_{O,A}^{(a)} = \begin{bmatrix} \mathbf{v}_{O,A}^{(a)} \\ \boldsymbol{\omega}_{A}^{(a)} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{\rho}}_{O,A} \\ \boldsymbol{\omega}_{A}^{(a)} \end{bmatrix} . \tag{12}$$

Vector-real **absolute** generalized velocity at the pole O_1:

$$\mathbf{u}_{O_1,A}^{(a)} = \begin{bmatrix} \mathbf{v}_{O_1,A}^{(a)} \\ \boldsymbol{\omega}_{A}^{(e)} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{\rho}}_{O_1,A} \\ \boldsymbol{\omega}_{A}^{(e)} \end{bmatrix} . \tag{13}$$

Vector-real **relative** generalized velocity at the pole O:

$$\mathbf{u}_{O,A}^{(r)} = \begin{bmatrix} \mathbf{v}_{O,A}^{(r)} \\ \boldsymbol{\omega}_{A}^{(r)} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{\delta}}_{O,A} \\ \boldsymbol{\omega}_{A}^{(r)} \end{bmatrix} . \tag{14}$$

Using the theorem for summation of angle velocities:

$$\boldsymbol{\omega}_{A}^{(a)} = \boldsymbol{\omega}_{A}^{(e)} + \boldsymbol{\omega}_{A}^{(r)} \;. \tag{15}$$

for the vector-real absolute generalized velocity at the pole O can be composed the following formula:

$$\mathbf{u}_{O,A}^{(a)} = \begin{bmatrix} \mathbf{v}_{O,A}^{(a)} \\ \boldsymbol{\omega}_{A}^{(a)} \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \boldsymbol{\Delta}_{O,A}^{T} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} . \begin{bmatrix} \mathbf{v}_{O_1,A}^{(a)} \\ \boldsymbol{\omega}_{A}^{(e)} \end{bmatrix} + \begin{bmatrix} \mathbf{v}_{O,A}^{(r)} \\ \boldsymbol{\omega}_{A}^{(r)} \end{bmatrix} , \tag{16}$$

$$\mathbf{E} = \mathbf{diag}\left[1\right]_3 . \tag{17}$$

The matrix $\mathbf{0}$ is a zero matrix, which has dimension 3×3.

Now, the following matrix is introduced:

$$\mathbf{Y}_{O,A} = \begin{bmatrix} \mathbf{E} & \boldsymbol{\Delta}_{O,A}^{T} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} . \tag{18}$$

Then formula (16) is written in a compact form as follows:

$$\mathbf{u}_{O,A}^{(a)} = \mathbf{Y}_{O,A} . \mathbf{u}_{O_1,A}^{(a)} + \mathbf{u}_{O,A}^{(r)} . \tag{19}$$

A vector-real **transmissive** generalized velocity at the pole O is defined:

$$\mathbf{u}_{O,A}^{(e)} = \mathbf{Y}_{O,A} . \mathbf{u}_{O_1,A}^{(a)} . \tag{20}$$

Using the theorem for summation of linear velocities:

$$\mathbf{v}_{O,A}^{(a)} = \mathbf{v}_{O,A}^{(e)} + \mathbf{v}_{O,A}^{(r)} , \tag{21}$$

and the theorem for summation of angle velocities, formula (15), a **new theorem** for summation of rigid body **generalized velocities** at the pole O is defined, namely:

$$\mathbf{u}_{O,A}^{(a)} = \mathbf{u}_{O,A}^{(e)} + \mathbf{u}_{O,A}^{(r)} . \tag{22}$$

This theorem, described by equation (22), is talked by the following manner: *"The vector-real **absolute** generalized velocity at the pole O is a vector sum of the vector-real **transmissive** generalized velocity and the vector-real **relative** generalized velocity determined at the same pole."*

For six-dimensional vectors and matrices, analogous six-dimensional block diagonal transition matrices are constructed:

$$\mathbf{W}_{A,B} = \begin{bmatrix} \mathbf{U}_{A,B} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{A,B} \end{bmatrix} = \mathbf{W}_{B,A}^{T} , \qquad \mathbf{W}_{B,A} = \begin{bmatrix} \mathbf{U}_{B,A} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{B,A} \end{bmatrix} = \mathbf{W}_{A,B}^{T} , \tag{23}$$

$$\mathbf{W}_{B,L} = \begin{bmatrix} \mathbf{U}_{B,L} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{B,L} \end{bmatrix} = \mathbf{W}_{L,A}^{T} , \qquad \mathbf{W}_{L,B} = \begin{bmatrix} \mathbf{U}_{L,B} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{L,B} \end{bmatrix} = \mathbf{W}_{B,L}^{T} , \tag{24}$$

$$\mathbf{W}_{A,L} = \mathbf{W}_{A,B} . \mathbf{W}_{B,L} = \mathbf{W}_{L,A}^{T} , \qquad \mathbf{W}_{L,A} = \mathbf{W}_{L,B} . \mathbf{W}_{B,A} = \mathbf{W}_{A,L}^{T} . \tag{25}$$

Using the theorem for summation of angle velocities, formula (15), the matrix-absolute angle velocity $\boldsymbol{\Omega}_{A}^{(e)}$ is determined as follows:

$$\boldsymbol{\Omega}_{A}^{(a)} = \boldsymbol{\Omega}_{A}^{(e)} + \boldsymbol{\Omega}_{A}^{(r)} . \tag{26}$$

The matrix $\boldsymbol{\Phi}_{A}^{(a)}$ has a block diagonal structure and it is composed by the formulas:

$$\boldsymbol{\Phi}_{A}^{(a)} = \begin{bmatrix} \boldsymbol{\Omega}_{A}^{(a)} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_{A}^{(a)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Omega}_{A}^{(e)} + \boldsymbol{\Omega}_{A}^{(r)} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_{A}^{(e)} + \boldsymbol{\Omega}_{A}^{(r)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Omega}_{A}^{(e)} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_{A}^{(e)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Omega}_{A}^{(r)} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}_{A}^{(r)} \end{bmatrix} , \tag{27}$$

$$\boldsymbol{\Phi}_{A}^{(a)} = \boldsymbol{\Phi}_{A}^{(e)} + \boldsymbol{\Phi}_{A}^{(r)} . \tag{28}$$

The formula (22) is differentiated towards the time:

$$\frac{d}{dt}\left(\mathbf{u}_{O,A}^{(a)} \right) = \frac{d}{dt}\left(\mathbf{u}_{O,A}^{(e)} \right) + \frac{d}{dt}\left(\mathbf{u}_{O,A}^{(r)} \right) , \tag{29}$$

$$\frac{d}{dt}\left(\mathbf{u}_{O,A}^{(a)} \right) = \frac{d}{dt}\left(\mathbf{Y}_{O,A} . \mathbf{u}_{O_1,A}^{(a)} \right) + \frac{d}{dt}\left(\mathbf{u}_{O,A}^{(r)} \right) . \tag{30}$$

The first derivative on the right-hand side of equation (30) has the following form:

$$\frac{d}{dt}\left(\mathbf{Y}_{O,A} . \mathbf{u}_{O_1,A}^{(a)} \right) = \frac{d}{dt}\left(\mathbf{W}_{A,B} . \mathbf{Y}_{O,B} . \mathbf{W}_{B,A} . \mathbf{u}_{O_1,A}^{(a)} \right) =$$

$$= \dot{\mathbf{W}}_{A,B} . \mathbf{Y}_{O,B} . \mathbf{W}_{B,A} . \mathbf{u}_{O_1,A}^{(a)} + \mathbf{W}_{A,B} . \dot{\mathbf{Y}}_{O,B} . \mathbf{W}_{B,A} . \mathbf{u}_{O_1,A}^{(a)} +$$

$$+ \mathbf{W}_{A,B} . \mathbf{Y}_{O,B} . \dot{\mathbf{W}}_{B,A} . \mathbf{u}_{O_1,A}^{(a)} + \mathbf{W}_{A,B} . \mathbf{Y}_{O,B} . \mathbf{W}_{B,A} . \dot{\mathbf{u}}_{O_1,A}^{(a)} =$$

$$= \mathbf{W}_{A,B}.\mathbf{\Phi}_B^{(e)}.\mathbf{Y}_{O,B}.\mathbf{W}_{B,A}.\mathbf{u}_{O_1,A}^{(a)} + \mathbf{W}_{A,B}.\dot{\mathbf{Y}}_{O,B}.\mathbf{W}_{B,A}.\mathbf{u}_{O_1,A}^{(a)} -$$
$$- \mathbf{W}_{A,B}.\mathbf{Y}_{O,B}.\mathbf{W}_{B,A}.\mathbf{\Phi}_A^{(e)}.\mathbf{u}_{O_1,A}^{(a)} + \mathbf{Y}_{O,A}.\dot{\mathbf{u}}_{O_1,A}^{(a)} =$$
$$= \mathbf{W}_{A,B}.\mathbf{\Phi}_B^{(e)}.\mathbf{W}_{B,A}.\mathbf{W}_{A,B}.\mathbf{Y}_{O,B}.\mathbf{W}_{B,A}.\mathbf{u}_{O_1,A}^{(a)} + \dot{\mathbf{Y}}_{O,A}.\mathbf{u}_{O_1,A}^{(a)} -$$
$$- \mathbf{Y}_{O,A}.\mathbf{\Phi}_A^{(e)}.\mathbf{u}_{O_1,A}^{(a)} + \mathbf{Y}_{O,A}.\dot{\mathbf{u}}_{O_1,A}^{(a)} =$$
$$= \left(\mathbf{\Phi}_A^{(e)}.\mathbf{Y}_{O,A} - \mathbf{Y}_{O,A}.\mathbf{\Phi}_A^{(e)} \right).\mathbf{u}_{O_1,A}^{(a)} + \dot{\mathbf{Y}}_{O,A}.\mathbf{u}_{O_1,A}^{(a)} + \mathbf{Y}_{O,A}.\dot{\mathbf{u}}_{O_1,A}^{(a)} \ . \qquad (31)$$

The second derivative on the right-hand side of equation (30) has the following form:

$$\frac{d}{dt}\left(\mathbf{u}_{O,A}^{(r)} \right) = \frac{d}{dt}\left(\mathbf{W}_{A,B}.\mathbf{W}_{B,L}.\mathbf{u}_{O,L}^{(r)} \right) =$$
$$= \dot{\mathbf{W}}_{A,B}.\mathbf{W}_{B,L}.\mathbf{u}_{O,L}^{(r)} + \mathbf{W}_{A,B}.\dot{\mathbf{W}}_{B,L}.\mathbf{u}_{O,L}^{(r)} + \mathbf{W}_{A,B}.\mathbf{W}_{B,L}.\dot{\mathbf{u}}_{O,L}^{(r)} =$$
$$= \mathbf{W}_{A,B}.\mathbf{\Phi}_B^{(e)}.\mathbf{W}_{B,L}.\mathbf{u}_{O,L}^{(r)} + \mathbf{W}_{A,B}.\mathbf{W}_{B,L}.\mathbf{\Phi}_L^{(r)}.\mathbf{u}_{O,L}^{(r)} + \dot{\mathbf{u}}_{O,A}^{(r)} =$$
$$= \mathbf{W}_{A,B}.\mathbf{\Phi}_B^{(e)}.\mathbf{W}_{B,A}.\mathbf{W}_{A,B}.\mathbf{u}_{O,B}^{(r)} + \mathbf{W}_{A,B}.\mathbf{W}_{B,L}.\mathbf{\Phi}_L^{(r)}.\mathbf{W}_{L,B}.\mathbf{W}_{B,L}.\mathbf{u}_{O,L}^{(r)} + \dot{\mathbf{u}}_{O,A}^{(r)} =$$
$$= \mathbf{\Phi}_A^{(e)}.\mathbf{u}_{O,A}^{(r)} + \mathbf{W}_{A,B}.\mathbf{\Phi}_B^{(r)}.\mathbf{u}_{O,B}^{(r)} + \dot{\mathbf{u}}_{O,A}^{(r)} =$$
$$= \mathbf{\Phi}_A^{(e)}.\mathbf{u}_{O,A}^{(r)} + \mathbf{W}_{A,B}.\mathbf{\Phi}_B^{(r)}.\mathbf{W}_{B,A}.\mathbf{W}_{A,B}.\mathbf{u}_{O,B}^{(r)} + \dot{\mathbf{u}}_{O,A}^{(r)} =$$
$$= \left(\mathbf{\Phi}_A^{(e)} + \mathbf{\Phi}_A^{(r)} \right).\mathbf{u}_{O,A}^{(r)} + \dot{\mathbf{u}}_{O,A}^{(r)} \ . \qquad (32)$$

The four **new generalized kinematic characteristics** are defined.

The first one is the vector-real **absolute** generalized acceleration at the pole O:

$$\boldsymbol{\alpha}_{O,A}^{(a)} = \frac{d}{dt}\left(\mathbf{u}_{O,A}^{(a)} \right) = \dot{\mathbf{u}}_{O,A}^{(a)} \ . \qquad (33)$$

The second one is the vector-real **transmissive** generalized acceleration at the pole O:

$$\boldsymbol{\alpha}_{O,A}^{(e)} = \left(\mathbf{\Phi}_A^{(e)}.\mathbf{Y}_{O,A} - \mathbf{Y}_{O,A}.\mathbf{\Phi}_A^{(e)} \right).\mathbf{u}_{O_1,A}^{(a)} + \mathbf{Y}_{O,A}.\dot{\mathbf{u}}_{O_1,A}^{(a)} \ . \qquad (34)$$

The third one is the vector-real **relative** generalized acceleration at the pole O:

$$\boldsymbol{\alpha}_{O,A}^{(r)} = \mathbf{\Phi}_A^{(r)}.\mathbf{u}_{O,A}^{(r)} + \dot{\mathbf{u}}_{O,A}^{(r)} \ . \qquad (35)$$

The forth one is a vector, which plays the same role as the Coriolis acceleration when the relative motion of a material point is studied, [2, 3, 4, 5]. This new generalized kinematic characteristic will be named the vector-real **Coriolisian** generalized acceleration at the pole O and it has the following form:

$$\boldsymbol{\alpha}_{O,A}^{(c)} = \dot{\mathbf{Y}}_{O,A}.\mathbf{u}_{O_1,A}^{(a)} + \mathbf{\Phi}_A^{(e)}.\mathbf{u}_{O,A}^{(r)} \ . \qquad (36)$$

Taking into account all equations from (29) till (36), a **new theorem** for summation of rigid body **generalized acceleration** at the pole O is defined, namely:

$$\boldsymbol{\alpha}_{O,A}^{(a)} = \boldsymbol{\alpha}_{O,A}^{(e)} + \boldsymbol{\alpha}_{O,A}^{(r)} + \boldsymbol{\alpha}_{O,A}^{(c)} \ . \qquad \mathbf{(37)}$$

This theorem, described by equation (37), is talked by the following manner: *"The vector-real **absolute** generalized acceleration at the pole O is a vector sum of the vector-real **transmissive** generalized acceleration, the vector-real **relative** generalized acceleration and the vector-real **Coriolisian** generalized acceleration determined at the same pole."*

IV. PURPOSE OF THE STUDY

The main purpose of this study is:

- Through the new theorem for summation of rigid body generalized velocities at the pole O,
- Through the new theorem for summation of rigid body generalized acceleration at the pole O,
- Through the differential equations, which have been obtained in previous author's articles, describing the absolute general motion of a rigid body,

to be obtained in matrix form the differential equations describing the relative motion of investigated asymmetrical rigid body at a pole, which do not coincide with its mass center, and then the rigid body absolute generalized coordinates to be obtained.

V. EQUATIONS OF RIGID BODY ABSOLUTE MOTION

The following system of differential equations, which is described the absolute general motion of an arbitrary asymmetrical rigid body at the pole O, is obtained in the article [11, 12], namely:

$$\mathbf{A}_{O,A}.\dot{\mathbf{u}}_{O,A}^{(a)} + \left(\mathbf{\Phi}_{A}^{(a)}.\mathbf{B}_{O,A} - \mathbf{B}_{O,A}.\mathbf{\Phi}_{A}^{(a)} + \dot{\overline{\mathbf{T}}}_{O,A}.\mathbf{A}_{C,A}.\mathbf{K}_{C,A}. \right).\mathbf{u}_{O,A}^{(a)} = \mathbf{Q}_{O,A} \ . \qquad (38)$$

The matrix $\mathbf{A}_{O,A}$ has the following block diagonal structure:

$$\mathbf{A}_{O,A} = \begin{bmatrix} \mathbf{M} & \mathbf{S}_{C,A}^{T} \\ \mathbf{S}_{C,A} & \mathbf{J}_{O,A} \end{bmatrix} . \qquad (39)$$

The matrix $\mathbf{M}$ is a diagonal and it is composed by the body mass m, namely:

$$\mathbf{M} = \mathbf{diag}\left[m \right]_3 \ . \qquad (40)$$

The matrix $\mathbf{J}_{O,A}$ is presented the rigid body inertia tensor for the pole O towards the coordinate system $O\,X\,Y\,Z$, namely:

$$\mathbf{J}_{O,A} = \begin{bmatrix} J_X & -J_{XY} & -J_{XZ} \\ -J_{YX} & J_Y & -J_{YZ} \\ -J_{ZX} & -J_{ZY} & J_Z \end{bmatrix} . \qquad (41)$$

The matrix $\mathbf{A}_{C,A}$ has also the block diagonal structure as follows:

$$\mathbf{A}_{C,A} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{C,A} \end{bmatrix} . \qquad (42)$$

The matrix $\mathbf{J}_{C,A}$ is presented the rigid body inertia tensor for the mass center C towards the coordinate system $C\,X_2 Y_2 Z_2$, which is not shown in the Fig. 1. This coordinate system has initial point the rigid body mass center C and its axis are parallel to the coordinate systems $O\,X\,Y\,Z$ and $N\,\xi\,\eta\,\zeta$.

The matrix $\mathbf{S}_{C,A}$ is presented the rigid body static tensor for the pole O towards the coordinate system $O\,X\,Y\,Z$, namely:

$$\mathbf{S}_{C,A} = m.\mathbf{R}_{C,A} \ , \qquad (43)$$

$$\mathbf{R}_{C,A} = \begin{bmatrix} 0 & -Z_C & Y_C \\ Z_C & 0 & -X_C \\ -Y_C & X_C & 0 \end{bmatrix} . \qquad (44)$$

The matrices $\overline{\mathbf{T}}_{O,A}$, $\mathbf{K}_{C,A}$ and $\mathbf{B}_{O,A}$ have the following structure:

$$\overline{\mathbf{T}}_{O,A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \tilde{\mathbf{R}}_{O,A} & \mathbf{0} \end{bmatrix} , \qquad (45)$$

$$\tilde{\mathbf{R}}_{O,A} = \begin{bmatrix} 0 & -\zeta_O & \eta_O \\ \zeta_O & 0 & -\xi_O \\ -\eta_O & \xi_O & 0 \end{bmatrix} , \qquad (46)$$

$$\mathbf{K}_{C,A} = \begin{bmatrix} \mathbf{E} & \mathbf{R}_{C,A}^{T} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} , \qquad (47)$$

$$\mathbf{B}_{O,A} = \begin{bmatrix} \mathbf{0} & -\mathbf{S}_{C,A}^{T} \\ \mathbf{S}_{C,A} & \mathbf{J}_{O,A} \end{bmatrix} . \qquad (48)$$

The vector $\mathbf{Q}_{O,A}$ is presented a vector-real generalized force of the rigid body for a pole O towards the coordinate system $O\,X\,Y\,Z$ and it contains the main force $\mathbf{F}_A$ and the main moment $\mathbf{M}_{O,A}$, determined by the reduction of all outer forces applying on the rigid body for that pole, namely:

$$\mathbf{Q}_{O,A} = \begin{bmatrix} \mathbf{F}_A \\ \mathbf{M}_{O,A} \end{bmatrix} . \qquad (49)$$

The formulas (22) and (37) are substituted in the differential equation (38), and then the following equation is obtained:

$$\mathbf{A}_{O,A}.\left(\boldsymbol{\alpha}_{O,A}^{(e)} + \boldsymbol{\alpha}_{O,A}^{(r)} + \boldsymbol{\alpha}_{O,A}^{(c)} \right) +$$

$$+\left(\mathbf{\Phi}_A^{(a)}.\mathbf{B}_{O,A} - \mathbf{B}_{O,A}.\mathbf{\Phi}_A^{(a)} + \dot{\overline{\mathbf{T}}}_{O,A}.\mathbf{A}_{C,A}.\mathbf{K}_{C,A}. \right).\left(\mathbf{u}_{O,A}^{(e)} + \mathbf{u}_{O,A}^{(r)} \right) = \mathbf{Q}_{O,A} \ . \tag{50}$$

Now, the formula (49) is substituted in the equation (50) as follows:

$$\begin{gathered}\mathbf{A}_{O,A}.\left(\mathbf{\alpha}_{O,A}^{(e)} + \mathbf{\alpha}_{O,A}^{(r)} + \mathbf{\alpha}_{O,A}^{(c)} \right) + \\ + \left(\mathbf{\Phi}_A^{(e)} + \mathbf{\Phi}_A^{(r)} \right).\mathbf{B}_{O,A}.\left(\mathbf{u}_{O,A}^{(e)} + \mathbf{u}_{O,A}^{(r)} \right) - \\ - \mathbf{B}_{O,A}.\left(\mathbf{\Phi}_A^{(e)} + \mathbf{\Phi}_A^{(r)} \right).\left(\mathbf{u}_{O,A}^{(e)} + \mathbf{u}_{O,A}^{(r)} \right) + \dot{\overline{\mathbf{T}}}_{O,A}.\mathbf{A}_{C,A}.\mathbf{K}_{C,A}.\left(\mathbf{u}_{O,A}^{(e)} + \mathbf{u}_{O,A}^{(r)} \right) = \mathbf{Q}_{O,A} \ .\end{gathered} \tag{51}$$

The matrix $\overline{\mathbf{T}}_{O,A}$ is presented as follows:

$$\overline{\mathbf{T}}_{O,A} = \overline{\mathbf{T}}_{O_1,A} + \mathbf{T}_{O,A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \widetilde{\mathbf{R}}_{O_1,A} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \Delta_{O,A} & \mathbf{0} \end{bmatrix} \ , \tag{52}$$

$$\widetilde{\mathbf{R}}_{O_1,A} = \begin{bmatrix} 0 & -\zeta_{O_1} & \eta_{O_1} \\ \zeta_{O_1} & 0 & -\xi_{O_1} \\ -\eta_{O_1} & \xi_{O_1} & 0 \end{bmatrix} . \tag{53}$$

The formula (52) is differentiated towards the time as follows:

$$\dot{\overline{\mathbf{T}}}_{O,A} = \dot{\overline{\mathbf{T}}}_{O_1,A} + \frac{d}{dt}\left(\mathbf{T}_{O,A} \right) = \dot{\overline{\mathbf{T}}}_{O_1,A} + \frac{d}{dt}\left(\mathbf{W}_{A,B}.\mathbf{W}_{B,L}.\mathbf{T}_{O,A}.\mathbf{W}_{L,B}.\mathbf{W}_{B,A} \right) . \tag{54}$$

The second derivative in the end of equation (54) is determined as follows:

$$\begin{gathered}\frac{d}{dt}\left(\mathbf{W}_{A,B}.\mathbf{W}_{B,L}.\mathbf{T}_{O,A}.\mathbf{W}_{L,B}.\mathbf{W}_{B,A} \right) = \\ = \dot{\mathbf{W}}_{A,B}.\mathbf{T}_{O,B}.\mathbf{W}_{B,A} + \mathbf{W}_{A,B}.\dot{\mathbf{W}}_{B,L}.\mathbf{T}_{O,L}.\mathbf{W}_{L,B}.\mathbf{W}_{B,A} + \\ = \mathbf{W}_{A,B}.\mathbf{W}_{B,L}.\dot{\mathbf{T}}_{O,L}.\mathbf{W}_{L,B}.\mathbf{W}_{B,A} + \mathbf{W}_{A,B}.\mathbf{W}_{B,L}.\mathbf{T}_{O,L}.\dot{\mathbf{W}}_{L,B}.\mathbf{W}_{B,A} + \mathbf{W}_{A,B}.\mathbf{T}_{O,B}.\dot{\mathbf{W}}_{B,A} = \\ = \mathbf{W}_{A,B}.\mathbf{\Phi}_B^{(e)}.\mathbf{T}_{O,B}.\mathbf{W}_{B,A} + \mathbf{W}_{A,B}.\mathbf{W}_{B,L}.\mathbf{\Phi}_L^{(r)}.\mathbf{T}_{O,L}.\mathbf{W}_{L,B}.\mathbf{W}_{B,A} + \\ + \dot{\mathbf{T}}_{O,A} - \mathbf{W}_{A,B}.\mathbf{W}_{B,L}.\mathbf{T}_{O,L}.\mathbf{W}_{L,B}.\mathbf{\Phi}_B^{(r)}.\mathbf{W}_{B,A} - \mathbf{W}_{A,B}.\mathbf{T}_{O,B}.\mathbf{W}_{B,A}.\mathbf{\Phi}_A^{(e)} = \\ = \mathbf{\Phi}_A^{(e)}.\mathbf{T}_{O,A} + \mathbf{\Phi}_A^{(r)}.\mathbf{T}_{O,A} + \dot{\mathbf{T}}_{O,A} - \mathbf{T}_{O,A}.\mathbf{\Phi}_A^{(r)} - \mathbf{T}_{O,A}.\mathbf{\Phi}_A^{(e)} = \\ = \left(\mathbf{\Phi}_A^{(e)} + \mathbf{\Phi}_A^{(r)} \right).\mathbf{T}_{O,A} - \mathbf{T}_{O,A}.\left(\mathbf{\Phi}_A^{(r)} + \mathbf{\Phi}_A^{(e)} \right) + \dot{\mathbf{T}}_{O,A} \ .\end{gathered} \tag{55}$$

The formula (55) is substituted in formula (54) and finally the following result is obtained:

$$\dot{\overline{\mathbf{T}}}_{O,A} = \dot{\overline{\mathbf{T}}}_{O_1,A} + \left(\mathbf{\Phi}_A^{(e)} + \mathbf{\Phi}_A^{(r)} \right).\mathbf{T}_{O,A} - \mathbf{T}_{O,A}.\left(\mathbf{\Phi}_A^{(r)} + \mathbf{\Phi}_A^{(e)} \right) + \dot{\mathbf{T}}_{O,A} \ . \tag{56}$$

Now, the formula (56) is substituted in the equation (51) as follows:

$$\begin{gathered}\mathbf{A}_{O,A}.\left(\mathbf{\alpha}_{O,A}^{(e)} + \mathbf{\alpha}_{O,A}^{(r)} + \mathbf{\alpha}_{O,A}^{(c)} \right) + \left(\mathbf{\Phi}_A^{(e)} + \mathbf{\Phi}_A^{(r)} \right).\mathbf{B}_{O,A}.\left(\mathbf{u}_{O,A}^{(e)} + \mathbf{u}_{O,A}^{(r)} \right) - \\ - \mathbf{B}_{O,A}.\left(\mathbf{\Phi}_A^{(e)} + \mathbf{\Phi}_A^{(r)} \right).\left(\mathbf{u}_{O,A}^{(e)} + \mathbf{u}_{O,A}^{(r)} \right) + \dot{\overline{\mathbf{T}}}_{O_1,A}.\mathbf{A}_{C,A}.\mathbf{K}_{C,A}.\left(\mathbf{u}_{O,A}^{(e)} + \mathbf{u}_{O,A}^{(r)} \right) + \\ + \left(\mathbf{\Phi}_A^{(e)} + \mathbf{\Phi}_A^{(r)} \right).\mathbf{T}_{O,A}.\mathbf{A}_{C,A}.\mathbf{K}_{C,A}.\left(\mathbf{u}_{O,A}^{(e)} + \mathbf{u}_{O,A}^{(r)} \right) - \\ - \mathbf{T}_{O,A}.\left(\mathbf{\Phi}_A^{(r)} + \mathbf{\Phi}_A^{(e)} \right).\mathbf{A}_{C,A}.\mathbf{K}_{C,A}.\left(\mathbf{u}_{O,A}^{(e)} + \mathbf{u}_{O,A}^{(r)} \right) + \\ + \dot{\mathbf{T}}_{O,A}.\mathbf{A}_{C,A}.\mathbf{K}_{C,A}.\left(\mathbf{u}_{O,A}^{(e)} + \mathbf{u}_{O,A}^{(r)} \right) = \mathbf{Q}_{O,A} \ .\end{gathered} \tag{57}$$

The resulting system of differential equations (57) contains the unknown functions $\delta_{O,X}$, $\delta_{O,Y}$, $\delta_{O,Z}$, ψ_r, θ_r, φ_r; $\dot{\delta}_{O,X}$, $\dot{\delta}_{O,Y}$, $\dot{\delta}_{O,Z}$, $\dot{\psi}_r$, $\dot{\theta}_r$, $\dot{\varphi}_r$; $\ddot{\delta}_{O,X}$, $\ddot{\delta}_{O,Y}$, $\ddot{\delta}_{O,Z}$, $\ddot{\psi}_r$, $\ddot{\theta}_r$ and $\ddot{\varphi}_r$. This system can be integrated numerically at corresponding initial conditions by appropriate mathematical programs, for example **MatLab**. Its integration in the time area gives us the law of relative motion of the studied rigid body L, namely $\delta_{O,X}$, $\delta_{O,Y}$, $\delta_{O,Z}$, ψ_r, θ_r and φ_r.

After determining of the vector-real **relative** generalized velocity, namely $\mathbf{u}_{O,A}^{(r)}$, using the formula (21), the vector-real **absolute** generalized velocity $\mathbf{u}_{O,A}^{(a)}$ can be obtained.

To find the law of absolute motion of the body L the following differential equation have to be solved:

$$\mathbf{H}.\dot{\mathbf{q}}^{(a)} = \mathbf{u}_{O,A}^{(a)} \ , \tag{58}$$

$$\mathbf{q}^{(a)} = \left\langle \begin{matrix} \xi_O & \eta_O & \zeta_O & \psi_a & \theta_a & \varphi_a \end{matrix} \right\rangle^T , \tag{59}$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{H}} \end{bmatrix} , \tag{60}$$

$$\overline{\mathbf{H}} = \begin{bmatrix} 1 & 0 & \sin\theta_a \\ 0 & \cos\psi_a & -\cos\theta_a .\sin\psi_a \\ 0 & \sin\psi_a & \cos\theta_a .\cos\psi_a \end{bmatrix} . \tag{61}$$

VI. CONCLUSION

Seven new additional kinematic characteristics are defined, namely, vectors-real generalized absolute, transmissive and relative velocity, vectors-real generalized absolute, transmissive, relative and Coriolisian acceleration. With these new additional kinematic characteristics, two new theorems are defined: the first one is for summation of the real generalized rigid body velocities, and the second one is for summation of the real generalized rigid body acceleration is formulated. A system of differential equations (57) in a matrix form is obtained, which serves to find the law of rigid body **relative** motion. Then, by means of other system of differential equations (58), the law of rigid body **absolute** motion is determined.

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REFERENCES

[1]. A. I. Ivanov, Theoretical Matrix Study of Rigid Body Relative Motion, *International Journal Of Advancement In Engineering Technology, Management and Applied Science, 05*(05), 2018, 21-28.
[2]. H. Goldstein, *Classical Mechanics* (Addison-Wesley, 2-nd edition, 1980).
[3]. D. Tong, *Classical Dynamics* (Wilberforce Road, UK, 2005).
[4]. A. Budo, *Theoretische Mechanik* (VEB Deutsher Verlag der Wissenschaften, Berlin, 1956).
[5]. G. Hamel, *Theoretische Mechanik* (Springer-Verlag, Berlin/Goetingen/Heidelberg, 1949).
[6]. Z. Meirovitch, *Methods of Analytical Mechanics* (McGraw-Hill Book Co., New York, 1970).
[7]. A. Cayley, A memoir on the theory of matrices, *Philosophical Transactions of the Royal Society of London, 148* , 1858, 17-37.
[8]. R. Featherstone, *Rigid body dynamics, Algorithms* (Springer Science plus Business Media, LLC, 2008).
[9]. V. Aslanov, *Rigid body dynamics for space applications* (Elsevier, Butterworth-Heinemann, 2017).
[10]. J. Strzalko, General motion of rigid body, *Lecture Notes Physics 792*, 2010, 23-39.
[11]. J. Wittenburg, *Dynamics of Multubody Systems* (Springer-Verlag, 1977).
[12]. A. Ivanov, Motion of asymmetrical rigid body in fluid area, *Annual of the University of Architecture, Civil Engineering and Geodesy, Sofia, 51*(2), 2017, 93–115.
[13]. A. Ivanov, Theoretical matrix study of rigid body general motion, *Greener Journal of Physics and Natural Sciences, 3*(2), 2017, 009-020.
[14]. F. P. Beer, E.R. Johnston, *Vector mechanics for engineers* (McGraw Hill, Inc., 1977).
[15]. E. Whittaker, *A treates on analytical dynamics of particles and rigid bodies*, (Cambridge Un. Press, 1959).

Anastas Ivanov Ivanov.” Theoretical Matrix Study of Rigid Body Absolute Motion."
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