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Generalization of Root Super multiplicities In Borcherds Superalgebras and corresponding Combinatorial Identities

K.Priyadharsini
PG and Research Department of Mathematics DG Vaishnav College, Chennai – 600 106
Corresponding Author: K.Priyadharsini

ABSTRACT
In this paper, root supermultiplicities and corresponding combinatorial identities for the Borcherds superalgebras which are extensions of A_n *and* B_n *were found out.*

KEYWORDS: *Borcherds superalgebras, Colored superalgebras, Root supermultiplicities.*
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I. INTRODUCTION

In 1977, the theory of Lie superalgebras was constructed by Kac. The theory of Lie superalgebras can also be seen in Scheunert(1979) in a detailed manner. Kac(1978) introduced the notion of Kac-Moody superalgebras and therein the Weyl-Kac character formula for the irreducible highest weight modules with dominant integral highest weight which yields a denominator identity when applied to 1-dimensional representation was also derived. A character formula called Weyl-Borcherds formula which yields a denominator identity for a generalized Kac-Moody algebra was proved in Borcherds(1988, 1992). Kang developed homological theory for the graded Lie algebras in 1993a and derived a closed form root multiplicity formula for all symmetrizable generalized Kac-Moody algebras in 1994a. The theory of generalized Lie superalgebra version of the generalized Kac-Moody algebras(Borcherds algebras) was introduced by Miyamoto(1996) and in that it was shown that the transformed Borcherds superalgebras and Borcherds superalgebras have many similar properties. Kim and Shin(1999) derived a recursive dimension formula for all graded Lie algebras. The dimension formula for graded Lie algebras was computed Kang and Kim(1999). Computation of root multiplicities of many Kac-Moody algebras and generalized Kac-Moody algebras can be seen in Frenkel and Kac(1980), Feingold and Frenkel(1983),Kass et al.(1990), Kang(1993b, 1994b,c,1996), Kac and Wakimoto(1994), Sthanumoorthy and Uma Maheswari(1996b), Hontz and Misra(2002), Sthanumoorthy et al.(2004a,b) and Sthanumoorthy and Lilly(2007b). Computation of root multiplicities of Borcherds superalgebras was found in Sthanumoorthy et al.(2009a). Some properties of different classes of root systems and their classifications for Kac-Moody algebras and Borcherds Kac-Moody algebras were studied in Sthanumoorthy and Uma Maheswari(1996a) and Sthanumoothy and Lilly(2000, 2002,2003,2004, 2007a). Also, properties of different root systems and complete classifications of special, strictly, purely imaginary roots of Borcherds Kac-Moody Lie superalgebras which are extensions of Kac-Moody Lie algebras were explained in Sthanumoorthy et al(2007,2009b) and Sthanumoorthy and Priyadharsini(2012, 2013).

Moreover, Kang(1998) obtained a superdimension formula for the homogeneous subspaces of the graded Lie superalgebras, which enabled one to study the structure of the graded Lie superalgebras in a unified way. Using the Weyl-Kac-Borcherds formula and the denominator identity for the Borcherds superalgerbas, Kang and Kim(1997) derived a dimension formula and combinatorial identities for the Borcherds superalgebras and found out the root multiplicities for Monstrous Lie superalgebras. Borcherds superalgebras which are extensions of Kac Moody algebras A_2 and A_3 were considered in Sthanumoorthy et al.(2009a) and therein dimension formulas were found out.In Sthanumoorthy and Priyadharsini(2014, 2015), we have computed combinatorial identities for A_2, A_3, B_2 and B_3 and root super multiplicities for hyperbolic Borcherds superalgebras. In Priyadharsini(2018), root supermultiplicities in Borcherds superalgebras A_4 and corresponding combinatorial identities were found.

In this paper, we compute root supermultiplicities and corresponding combinatorial identities for the Borcherds superalgebras which are extensions of Kac-Moody algebras A_n and B_n.

II. PRELIMINARIES

In this section, we give some basic concepts of Borcherds superalgebras as in Kang and Kim (1997).

Definition 2.1: Let I be a countable (possibly infinite) index set. A real square matrix $A=(a_{ij})_{i,j\in I}$ is called Borcherds-Cartan matrix if it satisfies:

(1) $a_{ii}=2$ or $a_{ii}\le 0$ for all $i\in I$,

(2) $a_{ij}\le 0$ if $i\ne j$ and $a_{ij}\in \mathbb{Z}$ if $a_{ii}=2$,

(3) $a_{ij}=0 \Leftrightarrow a_{ji}=0$

We say that an index i is real if $a_{ii}=2$ and imaginary if $a_{ii}\le 0$.

We denote by

$$I^{re}=\{i\in I \mid a_{ii}=2\},\ I^{im}=\{i\in I \mid a_{ii}\le 0\}.$$

Let $\underline{m}=\{m_i\in \mathbb{Z}_{>0} \mid i\in I\}$ be a collection of positive integer such that $m_i=1$ for all $i\in I^{re}$. We call $\underline{m}$, a charge of A.

Definition 2.2: A Borcherds-Cartan matrix A is said to be symmetrizable if there exists a diagonal matrix $D=\mathrm{d}iag(\varepsilon_i;i\in I)$ with $\varepsilon_i>0\ (i\in I)$ such that DA is symmetric.

Let $C=(c_{ij})_{i,j\in I}$ be a complex matrix satisfying $c_{ij}c_{ji}=1$ for all $i,j\in I$. Therefore, we have $c_{ii}=\pm 1$ for all $i\in I$. We call $i\in I$ an even index if $c_{ii}=1$ and an odd index if $c_{ii}=-1$.

We denote by I^{even} (I^{odd}), the set of all even (odd) indices.

Definition 2.3: A Borcherds-Cartan matrix $A=(a_{ij})_{i,j\in I}$ is restricted (or colored) with respect to C if it satisfies: If $a_{ii}=2$ and $c_{ii}=-1$ then a_{ij} are even integers for all $j\in I$. In this case, the matrix C is called a coloring matrix of A. Let $\mathfrak{h}=(\oplus_{i\in I}\mathbb{C}h_i)\oplus(\oplus_{i\in I}\mathbb{C}d_i)$ be a complex vector space with a basis $\{h_i,d_i;i\in I\}$, and for each $i\in I$, define a linear functional $\alpha_i\in\mathfrak{h}^\vee$ by

$$\alpha_i(h_j)=a_{ji},\ \alpha_i(d_j)=\delta_{ij}\ \ for\ all\ \ j\in I. \qquad (2.1)$$

If A is symmetrizable, then there exists a symmetric bilinear form $(.|.)$ on $\mathfrak{h}^\vee$ satisfying $(\alpha_i\mid\alpha_j)=\varepsilon_i a_{ij}=\varepsilon_j a_{ji}$ for all $i,j\in I$.

Definition 2.4: Let $Q=\oplus_{i\in I}\mathbb{Z}\alpha_i$ and $Q_+=\sum_{i\in I}\mathbb{Z}_{\ge 0}\alpha_i$, $Q_-=-Q_+$. Q is called the root lattice.

The root lattice Q becomes a (partially) ordered set by putting $\lambda\ge\mu$ if and only if $\lambda-\mu\in Q_+$.

The coloring matrix $C=(c_{ij})_{i,j\in I}$ defines a bimultiplicative form $\theta:Q\times Q\to C^X$ by

$$\theta(\alpha_i,\alpha_j)=c_{ij}\ \text{for all}\, i,j\in I,$$
$$\theta(\alpha+\beta,\gamma)=\theta(\alpha,\gamma)\theta(\beta,\gamma),$$
$$\theta(\alpha,\beta+\gamma)=\theta(\alpha,\beta)\theta(\alpha,\gamma)$$

for all $\alpha,\beta,\gamma\in Q$. Note that θ satisfies

$$\theta(\alpha,\beta)\theta(\beta,\alpha)=1\ \text{for}all\ \alpha,\beta\in Q, \qquad (2.2)$$

since $c_{ij}c_{ji}=1$ for all $i,j\in I$. In particular $\theta(\alpha,\alpha)=\pm 1$ for all $\alpha\in Q$.

We say $\alpha\in Q$ is even if $\theta(\alpha,\alpha)=1$ and odd if $\theta(\alpha,\alpha)=-1$.

Definition 2.5: A θ-colored Lie superalgebra is a Q-graded vector space $L=\oplus_{\alpha\in Q}L_\alpha$ together with a bilinear product $[,]:L\times L\to L$ satisfying

$$\lfloor L_\alpha,L_\beta\rfloor\subset L_{\alpha+\beta},$$
$$[x,y]=-\theta(\alpha,\beta)[y,x],$$

$$[x,[y,z]] = [[x,y],z] + \theta(\alpha,\beta)[y,[x,z]]$$

for all $\alpha, \beta \in Q$ and $x \in L_\alpha$, $y \in L_\beta, z \in L$.

In a θ-colored Lie superalgebra $L = \oplus_{\alpha\in Q} L_\alpha$, for $x \in L_\alpha$, we have $[x,x] = 0$ if α is even and $[x,[x,x]] = 0$ if α is odd.

Definition 2.6: The universal enveloping algebra $U(L)$ of a θ-colored Lie superalgebra L is defined to be $T(L)/J$, where $T(L)$ is the tensor algebra of L and J is the ideal of $T(L)$ generated by the elements $[x,y] - x\otimes y + \theta(\alpha,\beta) y \otimes x$ $(x \in L_\alpha, y \in L_\beta)$.

Definition 2.7: The Borcherds superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ associated with the symmetrizable Borcherds-Cartan matrix A of charge $\underline{m} = (m_i; i \in I)$ and the coloring matrix $C = (c_{ij})_{i,j\in I}$ is the θ-colored Lie superalgebra generated by the elements $h_i, d_i\,(i \in I), e_{ik}, f_{ik}\,(i \in I, k = 1,2,\cdots,m_i)$ with defining relations:

$$\begin{aligned}
[h_i, h_j] &= [h_i, d_j] = [d_i, d_j] = 0,\\
[h_i, e_{jl}] &= a_{ij} e_{jl}, [h_i, f_{jl}] = -a_{ij} f_{jl},\\
[d_i, e_{jl}] &= \delta_{ij} e_{jl}, [d_i, f_{jl}] = -\delta_{ij} f_{jl},\\
[e_{ik}, f_{jl}] &= \delta_{ij}\delta_{kl} h_i\\
(ade_{ik})^{1-a_{ij}} e_{jl} &= (adf_{ik})^{1-a_{ij}} f_{jl} = 0 \textit{ if } a_{ii} = 2 \textit{ and } i \neq j,\\
[e_{ik}, e_{jl}] &= [f_{ik}, f_{jl}] = 0 \textit{ if } a_{ij} = 0
\end{aligned}$$

for $i, j \in I, k = 1,\cdots,m_i, l = 1,\cdots,m_j$.

The abelian subalgebra $\mathfrak{h} = (\oplus_{i\in I} \mathbb{C} h_i) \oplus (\oplus_{i\in I} \mathbb{C} d_i)$ is called the Cartan subalgebra of $\mathfrak{g}$ and the linear functionals $\alpha_i \in \mathfrak{h}^\vee (i \in I)$ defined by (2.1) are called the simple roots of $\mathfrak{g}$. For each $i \in I^{re}$, let $r_i \in GL(\mathfrak{h}^\vee)$ be the simple reflection of $\mathfrak{h}^\vee$ defined by

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \quad (\lambda \in \mathfrak{h}^\vee).$$

The subgroup W of $GL(\mathfrak{h}^\vee)$ generated by the r_i's $(i \in I^{re})$ is called the Weyl group of the Borcherds super algebra $\mathfrak{g}$.

The Borcherds superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ has the root space decomposition $\mathfrak{g} = \oplus_{\alpha\in Q} \mathfrak{g}_\alpha$, where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \textit{ for all } h \in \mathfrak{h}\}.$$

Note that $\mathfrak{g}_{\alpha_i} = \mathbb{C} e_{i,1} \oplus \cdots \oplus \mathbb{C} e_{i,m_i}$

and $\mathfrak{g}_{-\alpha_i} = \mathbb{C} f_{i,1} \oplus \cdots \oplus \mathbb{C} f_{i,m_i}$

We say that $\alpha \in Q^X$ is a root of $\mathfrak{g}$ if $\mathfrak{g}_\alpha \neq 0$. The subspace $\mathfrak{g}_\alpha$ is called the root space of $\mathfrak{g}$ attached to α. A root α is called real if $(\alpha|\alpha) > 0$ and imaginary if $(\alpha|\alpha) \le 0$.

In particular, a simple root α_i is real if $a_{ii} = 2$ that is if $i \in I^{re}$ and imaginary if $a_{ii} \le 0$ that is if $i \in I^{im}$. Note that the imaginary simple roots may have multiplicity > 1. A root $\alpha > 0$ ($\alpha < 0$) is called positive (negative). One can show that all the roots are either positive or negative. We denote by Δ, Δ_+ and Δ_- the set of all roots, positive roots and negative roots, respectively. Also we denote $\Delta_{\bar{0}}$ ($\Delta_{\bar{1}}$) the set of all even (odd) roots of $\mathfrak{g}$. Define the subspaces $\mathfrak{g}^\pm = \oplus_{\alpha\in\Delta_\pm} \mathfrak{g}_\alpha$.

Then we have the triangular decomposition of $\mathfrak{g}$: $\quad \mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$.

Definition 2.8:(Sthanumoorthy et al.(2009b)) We define an indefinite nonhyperbolic Borcherds - Cartan matrix A, to be of extended-hyperbolic type, if every principal submatrix of A is of finite, affine, or hyperbolic type Borcherds - Cartan matrix. We say that the Borcherds superalgebra associated with a Borcherds - Cartan matrix A is of extended-hyperbolic type, if A is of extended-hyperbolic type.

Definition 2.9: A $\mathfrak{g}$-module V is called $\mathfrak{h}$-*diagonalizable*, if it admits a weight space decomposition $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$, where $V_\mu = \{v \in V \mid h\cdot v = \mu(h)v \, for\ all\ h \in \mathfrak{h}\}$.

If $V_\mu \neq 0$, then μ is called a *weight* of V, and $\dim V_\mu$ is called the multiplicity of μ in V.

Definition 2.10: A $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module V is called a *highest weight module* with highest weight $\lambda \in \mathfrak{h}^*$, if there is a nonzero vector $v_\lambda \in V$ such that

1. $e_{ik}\cdot v_\lambda = 0,$ for all $i \in I$, $k = 1,\cdots,m_i$,
2. $h\cdot v_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$,
3. $V = U(\mathfrak{g})\cdot v_\lambda$.

The vector v_λ is called a *highest weight vector*.

For a highest weight module V with highest weight λ, we have

(i) $V = U(\mathfrak{g}^-)\cdot v_\lambda$,

(ii) $V = \oplus_{\mu\le\lambda} V_\mu, V_\lambda = \mathbb{C}v_\lambda$ and

(iii) $\dim V_\mu < \infty$ for all $\mu \le \lambda$.

Definition 2.11: Let $P(V)$ denote the set of all weights of V. When all the weights spaces are finite dimensional, the character of V is defined to be $chV = \sum_{\mu\in\mathfrak{h}^\vee} (\dim V_\mu)e^\mu,$ where e^μ are the basis elements of the group $\mathbb{C}[\mathfrak{h}^\vee]$ with the multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for $\mu,\nu \in \mathfrak{h}^\vee$. Let $b_+ = \mathfrak{h}\oplus\mathfrak{g}_+$ be the Borel subalgebra of $\mathfrak{g}$ and $\mathbb{C}_\lambda$ be the 1-dimensional b_+-module defined by $\mathfrak{g}_+\cdot 1 = 0, h\cdot 1 = \lambda(h)1$ for all $h \in \mathfrak{h}$. The induced module $M(\lambda) = U(\mathfrak{g})\otimes_{U(b_+)} \mathbb{C}_\lambda$ is called the *Verma module* over $\mathfrak{g}$ with highest weight λ. Every highest weight $\mathfrak{g}$-module with highest weight λ is a homomorphic image of $M(\lambda)$ and the Verma module $M(\lambda)$ contains a unique maximal submodule $J(\lambda)$. Hence the quotient $V(\lambda) = M(\lambda)/J(\lambda)$ is irreducible.

Let P^+ be the set of all linear functionals $\lambda \in \mathfrak{h}^\vee$ satisfying

$$\begin{cases} \lambda(h_i) \in \mathbb{Z}_{\ge 0} & for\ all\ i \in I^{re} \\ \lambda(h_i) \in 2\mathbb{Z}_{\ge 0} & for\ all\ i \in I^{re}\cap I^{odd} \\ \lambda(h_i) \ge 0 & for\ all\ i \in I^{im} \end{cases}$$

The elements of P^+ are called the dominant integral weights.

Let $\rho \in \mathfrak{h}^\vee$ be the $\mathbb{C}$-linear functional satisfying $\rho(h_i) = \frac{1}{2}a_{ii}$ for all $i \in I$. Let T be the set of all imaginary simple roots counted with multiplicities, and for $F \subset T,$ we write $F \perp \lambda,$ if $\lambda(h_i) = 0$ for all $\alpha_i \in F.$

Definition 2.12:[Kang and Kim (1997)] Let J be a finite subset of I^{re} and we denote by $\Delta_J = \Delta \cap (\sum_{j\in J}\mathbb{Z}\alpha_j), \Delta_J^\pm = \Delta^\pm \cap \Delta_J$ and $\Delta^\pm(J) = \Delta^\pm \setminus \Delta_J^\pm.$ Let

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$$\mathfrak{g}_0^{(J)} = \mathfrak{h} \bigoplus \left(\bigoplus_{\alpha \in \Delta_J} \mathfrak{g}_\alpha \right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.3)$$

and

$$\mathfrak{g}_\pm^{(J)} = \bigoplus_{\alpha \in \Delta^\pm(J)} \mathfrak{g}_\alpha .$$

Then $\mathfrak{g}_0^{(J)}$ is the restricted Kac-Moody super algebra (with an extended Cartan subalgebra) associated with the Cartan matrix $A_J = (a_{ij})_{i,j \in J}$ and the set of odd indices $J^{odd} = J \cap I^{odd} = \{ j \in J \mid c_{jj} = -1 \}.$

Then the *triangular decomposition* of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{g}_-^{(J)} \bigoplus \mathfrak{g}_0^{(J)} \bigoplus \mathfrak{g}_+^{(J)}.$

Let $W_J = \langle r_j \mid j \in J \rangle$ be the subgroup of W generated by the simple reflections $r_j \; (j \in J),$ and let

$$W(J) = \left\{ w \in W \mid \Delta_w \subset \Delta^+(J) \right\}$$

where $\Delta_w = \{ \alpha \in \Delta^+ \mid w^{-1}\alpha < 0 \} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.4)$

Therefore W_J is the Weyl group of the restricted Kac-Moody super algebra $\mathfrak{g}_0^{(J)}$ and $W(J)$ is the set of right coset representatives of W_J in W. That is $W = W_J W(J).$

The following lemma given in Kang and Kim (1999), proved in Liu (1992), is very useful in actual computation of the elements of W(J).

Lemma 2.13: Suppose $w = w' r_j$ and $l(w) = l(w') + 1.$ Then $w \in W(J)$ if and only if $w' \in W(J)$ and $w'(\alpha_J) \in \Delta^+(J).$

Let $\Delta_{\bar{i},J}^\pm = \Delta_{\bar{i}}^\pm \cap \Delta_J \, (i = 0,1)$ and $\Delta_{\bar{i}}^\pm(J) = \Delta_{\bar{i}}^\pm \setminus \Delta_{\bar{i},J}^\pm \, (i = 0,1).$ Here $\Delta_{\bar{0}}^\pm (\Delta_{\bar{1}}^\pm)$ denotes the set of all positive or negative even (resp., positive or negative odd) roots of $\mathfrak{g}$.

The following proposition, proved in Kang and Kim (1997), gives the denominator identity for Borcherds superalgebras.

Proposition 2.14: [Kang and Kim (1997)]. Let J be a finite subset of the set of all real indices I^{re}. Then

$$\frac{\prod_{\alpha \in \Delta_{\bar{0}(J)}^-} (1 - e^\alpha)^{\dim \mathfrak{g}_\alpha}}{\prod_{\alpha \in \Delta_{\bar{1}(J)}^-} (1 + e^\alpha)^{\dim \mathfrak{g}_\alpha}} = \sum_{\substack{w \in W(J) \\ F \subset T}} (-1)^{l(w) + |F|} \, chV_J(w(\rho - s(F) - \rho))$$

where $V_J(\mu)$ denotes the irreducible highest weight module over the restricted Kac-Moody super algebra $\mathfrak{g}_0^{(J)}$ with highest weight μ and where F runs over all the finite subsets of T such that any two elements of F are mutually perpendicular. Here $l(w)$ denotes the length of $w, | F |$ the number of elements in F, and $s(F)$ the sum of the elements in F.

Definition 2.15: A basis elements of the group algebra $\mathbb{C}[\mathfrak{h}^\vee]$ by defining $E^\alpha = \theta(\alpha,\alpha) e^\alpha$. Also define the super dimension $Dim\mathfrak{g}_\alpha$ of the root space $\mathfrak{g}_\alpha$ by $Dim\mathfrak{g}_\alpha = \theta(\alpha,\alpha) \dim \mathfrak{g}_\alpha \ldots\ldots\ldots\ldots (2.5)$

Since $w(\rho - s(F)) - \rho$ is an element of Q_-, all the weights of the irreducible highest weight $\mathfrak{g}_0^{(J)}$-module $V_J(w(\rho - s(F)) - \rho)$ are also elements of Q_-. Hence one can define the super dimension $DimV_\mu$ of the weight space V_μ of $V_J(w(\rho - s(F)) - \rho)$ in a similar way. More generally, for an $\mathfrak{h}$-diagonalizable $\mathfrak{g}_0^{(J)}$-module $V = \oplus_{\mu \in \mathfrak{h}^\vee} V_\mu$ such that $P(V) \subset Q$, we define the super dimension $DimV_\mu$ of the weight space V_μ to be

$$DimV_{\mu} = \theta(\mu,\mu)\dim V_{\mu}..........(2.6)$$

For each $k \geq 1$, let

$$H_k^{(J)} = \bigoplus_{w\in W(J),F\subset T,l(w)+|F|=k} V_J(w(\rho - s(F)) - \rho)...............(2.7)$$

and define the homology space $H^{(J)}$ of $\mathfrak{g}_-^{(J)}$ to be

$$H^{(J)} = \sum_{k=1}^{\infty}(-1)^{k+1}H_k^{(J)} = H_1^{(J)}\Theta H_2^{(J)} \oplus H_3^{(J)}\Theta\cdots,.................(2.8)$$

an alternating direct sum of the vector spaces.

For $\tau \in Q_-$, define the super dimension $DimH_\tau^{(J)}$ of the τ -weight space of $H^{(J)}$ to be

$$\begin{aligned} DimH_\tau^{(J)} &= \sum_{k=1}^{\infty}(-1)^{k+1}(DimH_k^{(J)})_\tau \\ &= \sum_{k=1}^{\infty}(-1)^{k+1}\sum_{w\in W(J),F\subset T,l(w)+|F|=k} DimV_J(w(\rho - s(F)) - \rho)_\tau \\ &= \sum_{w\in W(J),F\subset T,l(w)+|F|\geq 1}(-1)^{l(w)+|F|+1}DimV_J(w(\rho - s(F)) - \rho)_\tau \qquad (2.9) \end{aligned}$$

Let

$$P(H^{(J)}) = \{\alpha \in Q^-(J) \mid \dim H_\alpha^{(J)} \neq 0\}..................(2.10)$$

and let $\{\tau_1, \tau_2, \tau_3, \cdots\}$, be an enumeration of the set $P(H^{(J)})$. Let $D(i) = DimH_{\tau_i}^{(J)}$.

Remark:

The elements of $P(H^{(J)})$ can be determined by applying the following proposition, proved in Kac (1990).

Proposition 2.16: (Kang and Kim, 1997)

Let $\Lambda \in P_+$. Then $P(\Lambda) = W.\{\lambda \in P_+ \mid \lambda \text{ is nondegenerate with respect to } \Lambda\}$.

For $\tau \in Q^-(J)$, define

$$T^{(J)}(\tau) = \{n = (n_i)_{i\geq 1} \mid n_i \in \mathbb{Z}_{\geq 0}, \sum n_i\tau_i = \tau\}.......................(2.11)$$

which is the set of all partitions of τ into a sum of α_i 's. For $n \in T^{(J)}(\alpha)$, use the notations $|n| = \sum n_i$ and $n! = \prod n_i!$.

Now, for $\tau \in Q^-(J)$, the *Witt partition function* $W^{(J)}(\tau)$ is defined as

$$W^{(J)}(\tau) = \sum_{n\in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!}\prod D(i)^{n_i}.......................(2.12)$$

Now a closed form formula for the super dimension $Dim\mathfrak{g}_\alpha$ of the root space $\mathfrak{g}_\alpha(\alpha \in \Delta^-(J))$ is given by the following theorem. The proof is given in Kang and Kim(1997).

Theorem 2.17: *Let J be a finite subset of I^{re}. Then, for $\alpha \in \Delta^-(J)$, we have*

$$Dim\mathfrak{g}_\alpha = \sum_{d|\alpha}\frac{1}{d}\mu(d)W^{(J)}\left(\frac{\alpha}{d}\right)$$

$$= \sum_{d|\alpha} \frac{1}{d} \mu(d) \sum_{n \in T^{(J)}\left(\frac{\alpha}{d}\right)} \frac{(|n|-1)!}{n!} \prod D(i)^{n_i} \text{...........}(2.13)$$

where μ is the classical Möbius function. Namely, for a natural number n, $\mu(n)$ is defined as follows:

$$\mu(n) = \begin{cases} 1 & for \quad n = 1, \\ (-1)^k & for \quad n = p_1 \cdots p_k \ (p_1, \cdots, p_k : distinct \ \ primes), \\ 0 & if \ it \ is \ not \ square \ free \end{cases}$$

and, for a positive integer d, $d \mid \alpha$ denotes $\alpha = d\alpha$ for some $\alpha \in Q_-$, in which case $\alpha = \frac{\alpha}{d}$.

In the following section III, we find root supermultiplicities of Borcherds superalgebras which are the Extensions of Kac-Moody Algebras with multiplicity r and the corresponding combinatorial identities using Kang and Kim(1997).

III. ROOT SUPERMULTIPLICITIES OF BORCHERDS SUPERALGEBRAS WHICH ARE THE EXTENSIONS OF KAC MOODY ALGEBRAS AND THE CORRESPONDING COMBINATORIAL IDENTITIES

3.1.Dimension Formula and combinatorial identity for the Borcherds superalgebra which is an extension of A_n

In this section, we find the superdimension Formula and combinatorial identity for the Borcherds superalgebra which is an extension of A_n using the same $J \subset \Pi^{re}$ and solving $T^{(J)}(\tau)$ in two different ways.

Consider the extended-hyperbolic Borcherds superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ associated with the extended-hyperbolic Borcherds-Cartan supermatrix A = $\begin{bmatrix} -k & -a_1 & -a_2 & \dots & \dots & \dots & -a_n \\ -b_1 & 2 & -1 & \cdots & \cdots & \cdots & 0 \\ -b_2 & -1 & 2 & -1 & \cdots & \cdots & 0 \\ \cdot & \vdots & -1 & 2 & \ddots & \ddots & -1 \\ \cdot & & & & & & \\ -b_n & 0 & \cdots & \cdots & \cdots & -1 & 2 \end{bmatrix}$ and the corresponding coloring matrix $C = \begin{bmatrix} -1 & c_{11} & c_{12} & \dots & \dots & \dots & c_{1n} \\ c_{11}^{-1} & 1 & c_{21} & \cdots & \cdots & \cdots & c_{2m} \\ c_{12}^{-1} & c_{21}^{-1} & 1 & \cdots & \cdots & \cdots & \cdots \\ \cdot & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \cdot & & & & & & \\ c_{1n}^{-1} & c_{2m}^{-1} & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}$ with all $c_{ij} \in C^X$.

Let $I = \{1,2,3,......,n\}$ be the index set with charge $\underline{m} = \{r,1,1,.....,1\}$.

Let us consider the root $\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + ... + k_n\alpha_n \in Q.$ We have $\theta(\alpha,\alpha) = (-1)^{k_1^2}$. Hence α is an even root(resp. odd root) if k_1 is even integer(resp. odd integer). Also $T = \{\alpha_1\}$ and the subset $F \subset T$ is either empty or $\{\alpha_1\}$. Take $J \subset \Pi^{re}$ as $J = \{2,3,4,5,...,n\}$. By Lemma 2.13, this implies that $W(J) = \{1\}$. From the equations (2.7) and (2.8), the homological space can be written as

$$H_1^{(J)} = V_J(1(\rho - \alpha_1) - \rho)$$
$$= V_J(-\alpha_1)$$
$$H_k^{(J)} = 0, \forall k \geq 2$$
$$Therefore H^{(J)} = H_1^{(J)} = V_J(-\alpha_1) \oplus V_J(-\alpha_1) \oplus ... \oplus V_J(-\alpha_1) \text{ (counted ‘r’ times)}$$

with

$$DimH^{(J)}_{(1,0,0,0,...,0)} = -r;\ DimH^{(J)}_{(1,1,0,0,....,0)} = -r;\ DimH^{(J)}_{(1,1,1,0,.....,0)} = -r;$$
$$DimH^{(J)}_{(1,1,1,1,......,0)} = -r;\ \cdots,\cdots,\cdots, DimH^{(J)}_{(1,1,1,1,.....,1,0)} = -r;\ DimH^{(J)}_{(1,1,1,...,1,a_1+1)} = -r;$$

Hence we have

$$P(H^{(J)}) = \{(1,0,0,0,....,0),(1,1,0,0,....,0),(1,1,1,0,...,0),(1,1,1,1,...,0),....,(1,1,1,1,...,1,0),(1,1,1,...,1,a_1+1)\}.$$

Let $\alpha = \tau = -p_1\alpha_1 - p_2\alpha_2 - p_3\alpha_3 - ...... - p_n\alpha_n \in Q_-$, with $(p_1, p_2, p_3,..., p_n) \in Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0} \times .... \times Z_{\geq 0}$.

Then by proposition.(2.16), we get

$$T^{(J)}(\tau) = \{(s_1, s_2, s_3, s_4,...., s_n, s_{n+1}) \mid s_1(1,0,0,0,...,0) + s_2(1,1,0,0,....,0) + s_3(1,1,1,0,....,0)$$
$$+ .... + s_{n+1}(1,1,1,.......,1,a_1+1) = (p_1, p_2, p_3, p_4,....., p_n)\}.$$

This implies $s_1 + s_2 + s_3 + ... + s_{n+1} = p_1$

$$s_2 + s_3 + ... + s_n + s_{n+1} = p_2$$
$$s_3 + s_4 + s_5 + ... + s_n + s_{n+1} = p_3$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ \ldots \ldots \ \ldots$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ \ldots \ldots \ \ldots$$
$$s_n + (a_1 + 1)s_{n+1} = p_n.$$

We have $s_1 = p_1 - p_2$;

$$s_2 = p_2 - p_3$$
$$s_3 = p_3 - p_4$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$s_{n-1} = p_{n-1} - p_n + a_1 s_{(n+1)}$$
$$s_n = p_n - (a_1+1)s_{n+1}$$
$$s_n = 0 \ to \ min(p_1, p_2, p_3,....,[\frac{p_n}{a_1+1}]).$$

Applying $s_1, s_2, s_3, s_4, s_5,...., s_{n+1}$ in Witt partition formula(eqn.2.12), we have

$$W^{(J)}(\tau) = \sum_{s_{n+1}=0}^{min(p_1,p_2,p_3,.....,[\frac{p_n}{a_1+1}])} \frac{(p_1-1)!(-r)^{p_1}}{(p_1-p_2)!(p_2-p_3)!(p_3-p_4)!....(p_{n-1}-p_n+a_1 s_{n+1})!(p_n-(a_1+1)s_{n+1})!s_{n+1}!} .(3.1.1)$$

The terms $(p_1 - p_2) \geq 0$, $(p_2 - p_3) \geq 0$, $(p_3 - p_4) \geq 0$, ……,$(p_{n-1} - p_{n+} a_1 s_{n+1}) \geq 0$,$(p_n-(a_1+1)s_{n+1}) \geq 0$ always. The terms in the denominator should be greater than or equal to zero.

From equation(2.13), the dimension g_α is

$$Dim g_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) W^{(J)}\left(\frac{\alpha}{d}\right)$$
$$= \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{n \in T^{(J)}\left(\frac{\alpha}{d}\right)} \frac{(|n|-1)!}{n!} \prod D(i)^{n_i},$$

Substituting the value of $W^{(J)}(\tau)$ from eqn. (3.1.1) in the above dimension formula, we have

$$Dim\mathfrak{g}_\alpha = \sum_{d|\alpha}\frac{1}{d}\mu(d) \sum_{s_{n+1}=0}^{min(p_1/d, p_2/d, p_3/d,.....[\frac{p_n}{d(a_1+1)}])} \frac{(\frac{p_1}{d}-1)!(-r)^{p_1/d}}{(\frac{p_1}{d}-\frac{p_2}{d})!(\frac{p_2}{d}-\frac{p_3}{d})!(\frac{p_3}{d}-\frac{p_4}{d})!...(\frac{p_{n-1}}{d}-\frac{p_n}{d}+\frac{a_1 s_{n+1}}{d})!(\frac{p_n}{d}-\frac{(a_1+1)s_{n+1}}{d})!\frac{s_{n+1}}{d}!}$$

If we solve the same $P(H^{(J)})$ using partition and substituting the partition in $T^{(J)}(\tau)$, we have

$$T^{(J)}(\tau) = \{p_1-p_2, p_2-p_3, p_3-p_4,......, p_{n-1}-|\phi|, |\phi|\},$$

where ϕ is partition of p_n with parts upto $(1, a_1+1)$ and of length less than or equal to p_n. That is, ϕ can be written as $(n_1, n_{(a_1+1)})$ where n_1 is number of 1's in the partition of 'p_n' and $n_{(a_1+1)}$ is number of (a_1+1)'s in the same partition of 'p_n'. Then $\phi = \{(p_n, 0), (p_n-(a_1+1), 1), ..., upto\, p_n - n_1(a_1+1) > 0\}$

Applying $T^{(J)}(\tau)$ in Witt partition formula (eqn.(2.12))

$$W^{(J)}(\tau) = \sum_{\phi\in T^{(J)}(\tau)} \frac{(p_1-1)!(-r)^{p_1}}{(p_1-p_2)!(p_2-p_3)!(p_3-p_4)!.....(p_{n-1}-|\phi|)|\phi|!} \quad ....(3.1.2)$$

From equation(2.13), the dimension $\mathfrak{g}_\alpha$ is

$$Dim\mathfrak{g}_\alpha = \sum_{d|\alpha}\frac{1}{d}\mu(d)W^{(J)}\left(\frac{\alpha}{d}\right)$$
$$= \sum_{d|\alpha}\frac{1}{d}\mu(d) \sum_{n\in T^{(J)}\left(\frac{\alpha}{d}\right)} \frac{(|n|-1)!}{n!}\prod D(i)^{n_i},$$

Substituting the value of $W^{(J)}(\tau)$ from eqn. (3.1.2) in the above dimension formula, we have

$$Dim\mathfrak{g}_\alpha = \sum_{d|\alpha}\frac{\mu(d)}{d} \sum_{\phi\in T^{(J)}(\tau)} \frac{(\frac{p_1}{d}-1)!(-1)^{\frac{p_1}{d}}}{(\frac{p_1}{d}-\frac{p_2}{d})!(\frac{p_2}{d}-\frac{p_3}{d})!(\frac{p_3}{d}-\frac{p_4}{d})!...(\frac{p_{n-1}}{d}-|\phi|)!|\phi|!}$$

Now consider the equation (3.1.1)

$$\sum_{(n_1, n_{(a_1+1)})} \frac{(p_1-1)!(-1)^{p_1}}{(p_1-p_2)!(p_2-p_3)!(p_3-p_4)!...(p_{n-1}-n_1-n_{(a_1+1)})!n_1!n_{(a_1+1)}!}$$

$$= \frac{(p_1-1)!(-r)^{p_1}}{(p_1-p_2)!(p_2-p_3)!......(p_{n-1}-p_n-0)!p_n!0!}$$
$$+\frac{(p_1-1)!(-r)^{p_1}}{(p_1-p_2)!(p_2-p_3)!(p_3-p_4)!...(p_{n-1}-p_n+a_1+1-1)!(p_n-(a_n+1))!1!}$$
$$+\frac{(p_1-1)!(-r)^{p_1}}{(p_1-p_2)!(p_2-p_3)!(p_3-p_4)!....(p_{n-1}-p_n+2a_1+2-2)!(p_n-2(a_n+1))!2!}$$

+ ……upto the term satisfying $p_n - n_1(a_1+1) > 0.$

$$= \sum_{\phi\in T^{(J)}(\tau)} \frac{(p_1-1)!(-r)^{p_1}}{(p_1-p_2)!(p_2-p_3)!(p_3-p_4)!......(p_{n-1}-|\phi|)!|\phi|!}.$$

where ϕ is partition of p_n with parts upto $(1, a_1+1)$ and of length p_n. Here we are considering the roots of type $(p_1, p_2, ..., p_n)$ such that $p_1 \geq p_i \leq p_n, i = 2,3,....,n-1, p_1 \geq p_n$. Hence we proved the following theorem.

Theorem 3.2.1: For the extended-hyperbolic Borcherds superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ associated with the extended-hyperbolic Borcherds-Cartan supermatrix A = $\begin{bmatrix} -k & -a_1 & -a_2 & \ldots & \ldots & \ldots & -a_n \\ -b_1 & 2 & -1 & \cdots & \cdots & \cdots & 0 \\ -b_2 & -1 & 2 & -1 & \cdots & \cdots & 0 \\ \cdot & \vdots & -1 & 2 & \ddots & \ddots & -1 \\ \cdot & & & & & & \\ -b_n & 0 & \cdots & \cdots & \cdots & -1 & 2 \end{bmatrix}$ let us consider $\alpha = \tau = -p_1\alpha_1 - p_2\alpha_2 - p_3\alpha_3 ...... - p_n\alpha_n \in Q_-$. Then the dimension of $\mathfrak{g}_\alpha$ is

$$Dim\mathfrak{g}_\alpha = \sum_{d|\alpha}\frac{1}{d}\mu(d) \sum_{s_{n+1}=0}^{min(p_1/d,\, p_2/d,\, p_3/d,.....[\frac{p_n}{d(a_1+1)}])} \frac{(\frac{p_1}{d}-1)!(-r)^{p_1/d}}{(\frac{p_1}{d}-\frac{p_2}{d})!(\frac{p_2}{d}-\frac{p_3}{d})!(\frac{p_3}{d}-\frac{p_4}{d})!..(\frac{p_{n-1}}{d}-\frac{p_n}{d}+\frac{a_1 s_{n+1}}{d})!(\frac{p_n}{d}-\frac{(a_1+1)s_{n+1}}{d})!\frac{s_{n+1}}{d}!}$$

Moreover, the following combinatorial identity holds:

$$\sum_{s_{n+1}=0}^{min(p_1,p_2,p_3,.....[\frac{p_n}{a_1+1}])} \frac{(p_1-1)!(-r)^{p_1}}{(p_1-p_2)!(p_2-p_3)!(p_3-p_4)!....(p_{n-1}-p_n+a_1 s_{n+1})!(p_n-(a_1+1)s_{n+1})!s_{n+1}!}$$

$$= \sum_{\phi\in T^{(J)}(\tau)} \frac{(p_1-1)!(-1)^{p_1}}{(p_1-p_2)!(p_2-p_3)!(p_3-p_4)!.......(p_{n-1}-|\phi|)!|\phi|!}.$$

where ϕ is partition of p_n with parts upto $(1, a_1+1)$ and of length p_n.

Example:

For the Borcherds Cartan matrix $A = \begin{pmatrix} -k & -1 & -b \\ -1 & 2 & -1 \\ -b & -1 & 2 \end{pmatrix}$, we consider the root τ = α = (5, 3, 4) ∈ Q−, r = 1, a_1 =1.

Substituting τ = α = (5,3,4) ∈ Q^- and a_1 = 1 in (3.1.1), we have

$$W^{(J)}(\tau) = \sum_{s_{n+1}=0}^{min(p_1,p_2,p_3,.....[\frac{p_n}{a_1+1}])} \frac{(p_1-1)!(-r)^{p_1}}{(p_1-p_2)!(p_2-p_3)!(p_3-p_4)!....(p_{n-1}-p_n+a_1 s_{n+1})!(p_n-(a_1+1)s_{n+1})!s_{n+1}!} ....(3.1.1)$$

Here $W^{(J)}(\tau) = \sum_{s_4=0}^{min(p_1,p_2,[\frac{p_3}{a_1+1}])} \frac{(p_1-1)!(-1)^{p_1}}{(p_1-p_2)!(p_2-p_3+a_1 s_4)!(p_3-(a_1+1)s_4)!s_4!}$

$$= \sum_{s_4=0}^{min(5,3,[\frac{4}{2}])} \frac{(5-1)!(-1)^5}{(5-3)!(3-4+s_4)!(4-2s_4)!s_4!} = \sum_{s_4=1}^{2} \frac{4!(-1)^5}{2!(-1+s_4)!(4-2s_4)!s_4!}$$

$$= \frac{4!(-1)}{2!1!0!2!} + \frac{4!(-1)}{2!1!0!2!} = -12.$$

Substituting τ = α = (5, 3, 4) ∈ Q^- and a_1 = 1 in (3.1.2), we have

$$W^{(J)}(\tau) = \sum_{\phi\in T^{(J)}(\tau)} \frac{(p_1-1)!(-1)^{p_1}}{(p_1-p_2)!(p_2-p_3)!(p_3-p_4)!.......(p_{n-1}-|\phi|)!|\phi|!}.$$

$$W^{(J)}(\tau) = \sum_{\phi\in T^{(J)}(\tau)} \frac{(p_1-1)!(-1)^{p_1}}{(p_1-p_2)!(p_2-|\phi|)!|\phi|!} = \frac{4!(-1)}{2!1!0!2!} + \frac{4!(-1)}{2!1!0!2!} = -12$$

3.2. Dimension Formula and combinatorial identity for the Borcherds superalgebra which is an extension of B_n.

Solving $T^{(J)}(\tau)$ in two different ways , here we are finding the superdimension Formula and combinatorial identity for the Borcherds superalgebra which is an extension of B_n using the same $J \subset \Pi^{re}$.

Consider the extended-hyperbolic Borcherds superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ associated with the extended-hyperbolic Borcherds-Cartan supermatrix A = $\begin{bmatrix} -k & -a_1 & -a_2 & \cdots & \cdots & \cdots & -a_n \\ -b_1 & 2 & -1 & \cdots & \cdots & \cdots & 0 \\ -b_2 & -1 & 2 & -1 & \cdots & \cdots & 0 \\ \cdot & \vdots & -1 & 2 & \ddots & \ddots & -1 \\ -b_n & 0 & \cdots & \cdots & \cdots & -2 & 2 \end{bmatrix}$ and the corresponding coloring matrix is $C = \begin{bmatrix} -1 & c_1 & c_2 & c_3 & c_4 \\ c_1^{-1} & 1 & c_5 & c_6 & c_7 \\ c_2^{-1} & c_5^{-1} & 1 & c_8 & c_9 \\ c_3^{-1} & c_6^{-1} & c_8^{-1} & 1 & c_{10} \\ c_4^{-1} & c_7^{-1} & c_9^{-1} & c_{10}^{-1} & 1 \end{bmatrix}$ with $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10} \in C^X$.

Let $I = \{1,2,3,....,n\}$ be the index set with charge $\underline{m} = \{r,1,1,1,1\}$.

Let us consider the root $\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + ... + k_n\alpha_n \in Q.$ We have $\theta(\alpha,\alpha) = (-1)^{k_1^2}$. Hence α is an even root(resp. odd root) if k_1 is even integer(resp. odd integer). Also $T = \{\alpha_1\}$ and the subset $F \subset T$ is either empty or $\{\alpha_1\}$. Take $J \subset \Pi^{re}$ as $J = \{3,4,5,...,n\}$. By Lemma 2.13, this implies that $W(J) = \{1, r_2\}$. From the equations (2.7) and (2.8), the homological space can be written as

$$H_1^{(J)} = V_J(1(\rho-\alpha_1)-\rho) \oplus V_J(r_2(\rho-\alpha_2)-\rho)$$
$$= V_J(-\alpha_1) \oplus V_J(-\alpha_2)$$
$$H_2^{(J)} = V_J(r_2(\rho-\alpha_1)-\rho) = V_J(-\alpha_1-(a_1+1)\alpha_2)$$
$$H_k^{(J)} = 0, \ \forall k \geq 3$$
$$Therefore H^{(J)} = V_J(-\alpha_1) \oplus V_J(-\alpha_2) \Theta V_J(-\alpha_1-(a_1+1)\alpha_2)$$ (counted 'r' times)

with

$$DimH^{(J)}_{(1,0,0,0,...,0)} = -r; \ DimH^{(J)}_{(0,1,0,0,....,0)} = -r; \ DimH^{(J)}_{(0,0,1,0,.....,0)} = -r;$$
$$DimH^{(J)}_{(0,0,0,1,......,0)} = -r; \ \cdots\cdots\cdots; \ DimH^{(J)}_{(1,1,1,...,1,a_1+1)} = -r;$$

Hence we have

$$P(H^{(J)}) = \{(1,0,0,0,....,0),(0,1,0,0,....,0),(0,0,1,0,...,0),....,(1,1,1,....,1),(1,1,1,...,1,a_1+1)\}.$$

Let $\tau = \alpha = -p_1\alpha_1 - p_2\alpha_2 - p_3\alpha_3 - ...... - p_n\alpha_n \in Q_-$, with $(p_1, p_2, p_3,..., p_n) \in Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0} \times .... \times Z_{\geq 0}$.

Then by proposition.(2.16), we get

$$T^{(J)}(\tau) = \{(s_1, s_2, s_3, s_4,...., s_n, s_{n+1}) \mid s_1(1,0,0,0,...,0) + s_2(0,1,0,0,....,0) + s_3(0,0,1,0,....,0)$$
$$+ .... + s_{n+1}(1,1,1,.......,1,a_n+1) = (p_1, p_2, p_3, p_4,....., p_n)\}.$$

This implies
$$s_1 + s_n + s_{n+1} = p_1$$
$$s_2 + s_n + s_{n+1} = p_2$$
$$s_3 + s_n + s_{n+1} = p_3$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots$$
$$s_n + (a_1+1)s_{n+1} = p_n.$$

We have
$$s_1 = p_1 - p_n + a_1 s_{n+1};$$
$$s_2 = p_2 - p_n + a_1 s_{n+1}$$
$$s_3 = p_3 - p_n + a_1 s_{n+1}$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$s_{n-1} = p_{n-1} - p_n + a_1 s_{(n+1)}$$
$$s_n = p_n - (a_1+1)s_{n+1}$$
$$s_{n+1} = 0 \; to \; min(p_1, p_2, p_3, \ldots, [\frac{p_n}{a_1+1}]).$$

Applying $s_1, s_2, s_3, s_4, s_5, \ldots, s_{n+1}$ in Witt partition formula(eqn.2.12), we have

$$W^{(J)}(\tau) = \sum_{s_{n+1}=0}^{min(p_1,p_2,p_3,\ldots,[\frac{p_n}{a_1+1}])} \frac{(P-(n-2)(p_n+a_1 s_{n+1})-1)!(-r)^{P-(n-2)(p_n+a_1 s_{n+1})}}{(p_1-p_n+a_1 s_{n+1})!\ldots(p_{n-1}-p_n+a_1 s_{n+!})!(p_n-(a_1+1)s_{n+1})!s_{n+1}!} \ldots.(3.2.1)$$

where $P = \sum P_i$, i = 1,2,…,n-1 and the terms in the denominator must be greater than or equal to zero.

From equation(2.13), the dimension g_α is

$$Dim g_\alpha = \sum_{d|\alpha} \frac{1}{d} \mu(d) W^{(J)}\left(\frac{\alpha}{d}\right)$$
$$= \sum_{d|\alpha} \frac{1}{d} \mu(d) \sum_{n \in T^{(J)}\left(\frac{\alpha}{d}\right)} \frac{(|n|-1)!}{n!} \prod D(i)^{n_i},$$

Substituting the value of $W^{(J)}(\tau)$ from eqn. (3.1.1) in the above dimension formula, we have

$$Dim g_\alpha = \sum_{d|\alpha} \frac{1}{d} \mu(d) \sum_{s_{n+1}=0}^{min(p_1,p_2,p_3,\ldots,[\frac{p_n}{a_1+1}])} \frac{(\frac{P}{d} - (n-2)\frac{p_n}{d} + (n-2)a_1 \frac{s_{n+1}}{d} - 1)!(-r)^{(P-(n-2)p_n+(n-2)a_1 s_{n+1})/d}}{(\frac{p_1}{d} - \frac{p_n}{d} + a_1\frac{s_{n+1}}{d})!\ldots(\frac{p_{n-1}}{d} - \frac{p_n}{d} + a_1 \frac{s_{n+1}}{d})!(\frac{p_n}{d} - (a_1+1)\frac{s_{n+1}}{d})!\frac{s_{n+1}}{d}!}$$

If we solve the same $P(H^{(J)})$ using partition and substituting the partition in T(J) (τ), we have

$$T^{(J)}(\tau) = \{p_1 - \phi_1, p_2 - \phi_1, \ldots, p_{n-1} - \phi_1, p_n - \phi_2, \phi_2\},$$

where ϕ_1 is the partition of p_n with parts $(1, a_1+1)$ of length p_n and ϕ_2 is the partition of s_4 with parts upto $\min(p_1, p_2, \ldots, \left[\frac{p_n}{a_1+1}\right])$.

Applying $T^{(J)}(\tau)$ in Witt partition formula (eqn.(2.12))

$$W^{(J)}(\tau) = \sum_{\phi \in T^{(J)}(\tau)} \frac{(P-(n-2)\phi_1 - 1)!(-r)^{P-(n-2)\phi_1}}{(p_1-\phi_1)!(p_2-\phi_1)!\ldots(p_{n-1}-\phi_1)!(p_n-\phi_2)!\phi_2!} \ldots\ldots\ldots\ldots(3.2.2)$$

where $P = \sum P_i$, i = 1,2,…,n-1, ϕ_1 is the partition of p_n with parts $(1, a_1+1)$ of length p_n and ϕ_2 is the partition of s_4 with parts upto $\min(p_1, p_2, \ldots, \left[\frac{p_n}{a_1+1}\right])$.

From equation(2.13), the dimension g_α is

$$Dim g_\alpha = \sum_{d|\alpha} \frac{1}{d} \mu(d) W^{(J)}\left(\frac{\alpha}{d}\right)$$
$$= \sum_{d|\alpha} \frac{1}{d} \mu(d) \sum_{n \in T^{(J)}\left(\frac{\alpha}{d}\right)} \frac{(|n|-1)!}{n!} \prod D(i)^{n_i},$$

Substituting the value of $W^{(J)}(\tau)$ from eqn. (3.1.2) in the above dimension formula, we have

$$Dim g_\alpha = \sum_{d|\alpha} \frac{\mu(d)}{d} \sum_{\phi \in T^{(J)}(\tau)} \frac{(\frac{P}{d} - (n-2)\frac{\phi_1}{d} - 1)!(-r)^{(P-(n-2)\phi_1)/d}}{(\frac{p_1}{d} - \frac{\phi_1}{d})!(\frac{p_2}{d} - \frac{\phi_1}{d})!\ldots(\frac{p_{n-1}}{d} - \frac{\phi_1}{d})!(\frac{p_n}{d} - \frac{|\phi_2|}{d})!\frac{|\phi_2|}{d}!}$$

where $P = \sum P_i$, i = 1,2,…,n-1, ϕ_1 is the partition of p_n with parts $(1, a_1+1)$ of length p_n and ϕ_2 is the partition of s_4 with parts upto $\min(p_1, p_2, \ldots, \left[\frac{p_n}{a_1+1}\right])$.

Theorem 3.2.1: For the extended-hyperbolic Borcherds superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ associated with the extended-hyperbolic Borcherds-Cartan supermatrix A = $\begin{bmatrix} -k & -a_1 & -a_2 & \ldots & \ldots & \ldots & -a_n \\ -b_1 & 2 & -1 & \cdots & \cdots & \cdots & 0 \\ -b_2 & -1 & 2 & -1 & \cdots & \cdots & 0 \\ \cdot & \vdots & -1 & 2 & \ddots & \ddots & -1 \\ -b_n & 0 & \cdots & \cdots & \cdots & -2 & 2 \end{bmatrix}$, let us consider $\alpha = \tau = -p\alpha_1 - q\alpha_2 - u\alpha_3 - v\alpha_4 - w\alpha_5 \in Q_-$. Then the dimension of $\mathfrak{g}_\alpha$ is

$$Dim\mathfrak{g}_\alpha = \sum_{d|\alpha} \frac{\mu(d)}{d} \sum_{s_{n+1}=0}^{min(p_1,p_2,p_3,\ldots..[\frac{p_n}{a_1+1}])} \frac{(P-(n-2)(p_n+a_1s_{n+1})-1)!(-r)^{P-(n-2)(p_n+a_1s_{n+1})}}{(p_1-p_n+a_1s_{n+1})!\ldots(p_{n-1}-p_n+a_1s_{n+!})!(p_n-(a_1+1)s_{n+1})!s_{n+1}!}.$$

Moreover, the following combinatorial identity holds:

$$\sum_{s_6=0}^{min(p_1,p_2,p_3,\ldots..[\frac{p_n}{a_n+1}])} \frac{(P-(n-2)(p_n+a_1s_{n+1})-1)!(-r)^{P-(n-2)(p_n+a_1s_{n+1})}}{(p_1-p_n+a_1s_{n+1})!\ldots(p_{n-1}-p_n+a_1s_{n+!})!(p_n-(a_1+1)s_{n+1})!s_{n+1}!}$$

$$= \sum_{\phi\in T^{(J)}(\tau)} \frac{(P-(n-2)\phi_1-1)!(-r)^{P-(n-2)\phi_1}}{(p_1-\phi_1)!(p_2-\phi_1)!\ldots.(p_{n-1}-\phi_1)!(p_n-\phi_2)!\phi_2!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.2.3)$$

where $P = \sum P_i$, i = 1,2,…,n-1, ϕ_1 is the partition of p_n with parts $(1, a_1+1)$ of length p_n and ϕ_2 is the partition of s_4 with parts upto $\min(p_1, p_2, \ldots, \left[\frac{p_n}{a_1+1}\right])$.

Example 3.2.2:

For the Borcherds-Cartan supermatrix A = $\begin{pmatrix} -k & -a & -b \\ -a & 2 & -1 \\ -b & -2 & 2 \end{pmatrix}$, consider a root α = τ = (5,4,3) with a=1. Applying in (3.2.1), we get

$$\sum_{s_6=0}^{min(p_1,p_2,p_3,\ldots..[\frac{p_n}{a_n+1}])} \frac{(P-(n-2)p_n+(n-2)a_1s_{n+1}-1)!(-r)^{P-(n-2)p_n+(n-2)a_1s_{n+1}}}{(p_1-p_n+a_1s_{n+1})!\ldots(p_{n-1}-p_n+a_1s_{n+!})!(p_n-(a_1+1)s_{n+1})!s_{n+1}!}$$

Here

$$W^{(J)}(\tau) = \sum_{s_4=0}^{\min(5,4,\left[\frac{3}{2}\right])} \frac{(5+4-3+s_4-1)!(-1)^{5+s_4}}{(5-3+s_4)!(4-3+s_4)!(3-2s_4)!s_4!}$$

$$= \frac{5!(-1)^5}{2!1!3!0!} + \frac{6!(-1)^6}{3!2!1!1!} = 50.$$

Substituting α = τ = (5, 3, 4), r = 2, a = 1 in (3.2.2), we have

$$\sum_{\phi\in T^{(J)}(\tau)} \frac{(P-(n-2)\phi_1-1)!(-r)^{P-(n-2)\phi_1}}{(p_1-\phi_1)!(p_2-\phi_1)!\ldots.(p_{n-1}-\phi_1)!(p_n-\phi_2)!\phi_2!}$$

$$= \frac{5!(-1)^5}{2!1!3!0!} + \frac{6!(-1)^6}{3!2!1!1!} = 50.$$

Hence the equality (3.2.3) holds.

IV. CONCLUSION

We conclude that for the extended-hyperbolic Borcherds superalgebras associated with the extended-hyperbolic Borcherds-Cartan supermatrices A_n and B_n, root supermultiplicities and corresponding combinatorial identities holds for the roots satisfying the conditions given in 3.1 and 3.2.

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