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On Covariance Of The Exponential Fractional Brownian Motion Of The Nigerian All-Share Index (1990-2007)

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--*ABSTRACT:*--
The stochastic dynamics and behaviour of the Nigerian Stock Market (NSM) as a whole are characterised by the daily movements of the Nigerian All Share Index (NASI). By using the Rescaled Range Statistics (R\S) to classify the time series, the Hurst parameter, H, was obtained in order to characterise the NSM in the form of an exponential fractional Brownian motion (fBm). The covariance, $R_H(t,s)$*, of the exponential fBm of the NASI was obtained in an unbiased long range dependence (LRD) phenomenon with H > 0.5.*
KEYWORDS: *Nigerian All-Share Index, Hurst Parameter, Covariance, Exponential Fractional Brownian motion, Long Range Dependence.*

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I. NIGERIAN STOCK EXCHANGE

The Nigerian Stock Exchange (NSE) was established in 1960 as the Lagos Stock Exchange. The NSE started its operations in Lagos in 1961 with 19 securities listed for trading while it became known as the Nigerian Stock Exchange in December 1977, with branches established in some major commercial cities of the country. The NSE maintains an All-Share Index formulated in January 3,1984. Only common stocks(ordinary shares) are included in the daily computation of the value-weighted index. NSE introduced the NSE-30 index, which is a sample-based capitalization-weighted index plus four sectoral indices. Example of some sectoral indices include: NSE Food/Beverages Index (now known as NSE Consumer Goods Index), NSE Banking Index, NSE Insurance Index, NSE Industrial Index and NSE

Oil & Gas Index. All listings are included in the Nigerian Stock Exchange All-share index. Data on listed companies performances are published daily, weekly, monthly, quarterly and annually. "Retrieved April 28, 2014, from NSE history.htm(en.wikipedia.org/wiki/Nigerian Stock Exchange)". In the work by [1], the NASI data (1990-2007) was analysed over 72 observations.

1.1 Stock Market Index

An Index is a statistical measure of change in an economy or a security market. In the case of financial markets, an index is an imaginary portfolio of securities representing a particular market or a portion of it. Each index has its own calculation methodology which is usually expressed in terms of a change from a base value. Thus the percentage change is more important than the actual numerical value.

Stocks and bond market indexes are used to construct mutual funds and exchange-traded funds (ETFs) whose portfolios reflect the components of the index. All NSE Indices are based on Market Capitalization methodology such that each stock represented in the index contributes to the index proportionally to its market capitalization. Only fully paid common shares, denominated in the Nigerian Currency are included in the index. Preferred shares, convertibles, bonds and mutual funds are all excluded.

The NSE-30, NSE-50 and NSE Industrial Indices are modified market capi- talization index. These are CAPPED indices (NSE Index Methodology.pdf 2013).

A stock market or equity market is the aggregation of buyers and sellers (that is, a loose network of economic transactions, not a physical facility or discrete entity) of stocks or shares which are securities on a Stock Exchange as well as those traded privately (stockmarket − wow.com.htm). According to Collins Cobuild (2003), a stock market consists of the activity of buying stocks and shares, and the people and institutions that organized it.

The Stocks are picked based on their market capitalization from the most liquid sectors. The liquidity is based on the number of times the stock is traded during the preceding two quarters. To be included, the stock must be traded for at least 70 percent of the number of times the market opened for business.

1.2 Calculation of a Stock Market Index

1.2.1 Market Capitalization Weighted Methodology

We present the adjusted market capitalization weighted methodology for calculating the NSE indices. The daily index value is calculated by dividing the Current Market Value (Closing price x Number of listed shares x Capping factor) of all constituent companies by a Base Market Value. Calculating the Base market value (BMV) as at the base date, one would divide the current market value (CMV) of all constituent companies by itself and multiply by 1000.

The NSE indices are calculated using the formula below:

$$\frac{Current\ Market\ Value}{Base\ Market\ Value}\times 1000 \text{ or } \frac{CMV}{BMV}\times 1000$$

where: CMV represented Current Market Value after the change and BMV represented Base Market Value after the change. Also,

$$\sum_{i=1}^{n} P_{a_i} Q_{a_i} = \sum_{i=1}^{n} P_{b_i} Q_{b_i} \times 1000$$

where : $P_{a_i} =$ Current market price of an ordinary share as at the base; $Q_{a_i} =$ Current number of listed ordinary shares; $P_{b_i} =$ Market price of an ordinary share as at the base date and

$Q_{b_i} =$ Number of listed shares as at the base date, i = 1,2,…,n, n = Number of constituents in the index. Where changes other than price variations occur which affect the index, an adjust-ment is made in order to eradicate the effects of such changes. Such adjustment is designed to make the index, after the changes equal to the index before the changes. The changes envisaged here include new listings, de-listings, and increase in the issued capital of listed companies. The procedure for effecting the adjustment is as follows:

$$\frac{CMV}{BMV} = \frac{CMV_0}{BMV_0}$$

1.3 Hurst Parameter

Hurst (1951) gave a long term storage capacity of water reservoirs. The Hurst parameter, H∈(0,1) is an estimator (without dimension) for the self-similarity of a times series initially defined by Harold Edwin Hurst to develop a law for regularities of the Nile's river water level, which had useful application in medicine and finance. In order to characterize the historical market's trend of NASI as contained [1], [3] published the value of the Hurst parameter as H = 0.78 > 0.5.

1.4 Autoregressive fractional integrated moving average (ARFIMA) and fBm

ARFIMA models may give rise to persistent or anti-persistent behaviour similar to fractional noise. The stochastic fractal processes, that is, fractional Brownian motion (fBm) had been discussed by Mandelbrot and Van Ness (1968). In fact, each of the fBm of Mandelbrot and Wallis (1969a,1969b,1969c,1969d) is an example of an ARFIMA(0,d,0) process. Since the more general ARFIMA(p,d,q) process can include short memory autoregressive (AR) or moving average (MA) processes over a long memory process, it has potential in describing markets. Some issues concerning fBm are considered in Duncan, et al. (2000), Coeurjolly (2000,2001,2013), Biagini et al. (2000,2008), Biagini and Øksendal (2003), Hu and Øksendal (2003), Mishura and Nualart (2003), and Decreusefond (2003), where minimal hedging of variance, stochastic calculus, application to fractional white noise calculus, finance and stochastic integration were discussed respectively. Also, Biagini et al. (2008) gave some approximations of stochastic integrals having fBm with H > 0.5. The original Black-Scholes formula, fBm and applications of Wick-Itö stochastic calculus in finance were treated in the book by Choi (2008). Due to financial market dynamics, some effects of autocorrelation function and partial autocorrelation functions and their corresponding residuals of the given NASI data were part of the R\S analysis results obtained in [1] and respectively published [2] and [4].

II. FRACTIONAL BROWNIAN MOTION AND WIENER INTEGRALS

2.1 Fractional Brownian Motion (fBm)

Fractional Brownian motion, as a stochastic process B_t^H, has one parameter $H,\ H \in (0,1)$, which can be characterized by its self-similarity. If we consider a scale parameter, $a > 0$, the process $(B_{t+a\tau}^H - B_\tau^H)$

has the same moments as $(a^H B_t^H)$. It is a non-stationary process, that is, its covariance function is a function of time, t, such that

$$E[B_t^H B_s^H] = \tfrac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}]$$

(2.1)

fBm has been of interest in graphics because as a first approximation it is a useful model of terrain. The main advantage of fBm as a model of terrain is a remarkable compactness of representation. Depending on how much deterministic data is included, the data base can be from two numbers to a few hundreds, to represent terrain that ultimately contains thousands or million of polygons. The second big advantage, due to its fractal nature, is that unlimited amount of details can be generated. The disadvantages include the fact that to generate a surface, pure recursive subdivision is not sufficient which will also complicate the subdivision algorithm. Also, it has limited flexibility with basically only one parameter to be adjusted to generate different terrains.

There have been numerous methods published for stochastic subdivisions. The criteria to evaluate them must depend on what they claim to accomplish and their intended use. The techniques to approximate specifically fractional Brownian motion have been well described by Fourier et al. (1982) where they put their methods into the context of computer graphics, and emphasized the problems of integrating the generation of approximations to fBm with traditional graphics modelling Fournier (1995).

2.2 Intrinsic Properties of fractional Brownian motion

Definition 2.1 Let $H \in (0,1)$. An fBm with H, is a continuous and centred Gaussian process with covariance function

$$E[B_t^H B_s^H] = \tfrac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}]$$

For $H = \dfrac{1}{2}$, the fBm is then a standard Brownian motion. The intrinsic properties of the standard fBm include:-

I. $B_0^H = 0$ and $E[B_t^H] = 0, \quad \forall\, t \ge 0$

II. B_t^H has homogeneous increments, i.e. $B_{t-s}^H - B_s^H$ has the same law as B_t^H, $\forall\, s,t \ge 0.$

III. B_t^H is a Gaussian process, for all $H \in (0,1).$

IV. B_t^H has continuous trajectories.

2.3 Long-Range Dependence

Definition 2.2 A stationary sequence $(X_n)_{n\in\mathbb{N}}$ exhibits LRD if the ACF $\tau(n) := \mathrm{cov}(X_k, X_{k+n})$ satisfies

$$\lim_{n\to\infty} \frac{\tau(n)}{cn^{-\alpha}} = 1 \qquad (2.2)$$

for some constant c and $\alpha \in (0,1).$ In this case, the dependence between X_k and X_{k+n} decays slowly as $n \to \infty$ and

$$\sum_{n=1}^{\infty} \tau(n) = \infty \qquad (2.3)$$

According to Biagini et al.(2008:p.9), the increments

$$X_k := B_k^H - B_{k-1}^H \qquad (2.4)$$

and

$$X_{k+n} := B_{k+n}^H - B_{k-1}^H \qquad (2.5)$$

are LRD if $H > \dfrac{1}{2}$, where

$$\tau_H(n) = \frac{1}{2}[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}] \sim H(2H-1)n^{2H-2}, n \to \infty \qquad (2.6)$$

Also,

$$\lim_{n\to\infty}\frac{\tau_H(n)}{H(2H-1)n}=1 \tag{2.7}$$

Summarizing, we obtain:

$$\sum_{n=1}^{\infty}\tau_H(n)=\infty,\quad H>\frac{1}{2} \tag{2.8}$$

$$\sum_{n=1}^{\infty}\left|\tau_H(n)\right|<\infty,\quad H<\frac{1}{2} \tag{2.9}$$

The spectral density of the autocovariance $\tau(k)$ is given by

$$f(\lambda):=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}e^{-i\lambda k}\tau(k)\text{ for }\lambda\in[-\pi,\pi] \tag{2.10}$$

Alternative definitions of LRD are given below.

Definition 2.3 For stationary sequences, , with finite variance, we say that $(X_n)_{n\in\mathbb{N}}$ exhibits long-range dependence if one of the following holds:

For some constant c and $\beta\in(0,1)$,

$$\lim_{n\to\infty}\sum_{k=-n}^{\infty}\tau(k)\big/cn^{\beta}L_1(n)=1 \tag{2.11a}$$

For some constant c and $\gamma\in(0,1)$,

$$\lim_{k\to\infty}\tau(k)\big/ck^{-\gamma}L_2(k)=1 \tag{2.11b}$$

For some constant c and $\delta\in(0,1)$,

$$\lim_{\lambda\to 0}f(\lambda)\big/c|\lambda|^{-\delta}L_3(|\lambda|)=1 \tag{2.11c}$$

where L_1, L_2 are slowly varying functions at infinity, while L_3 is slowly varying at zero.

Lemma 2.1. For fBm, B_t^H of Hurst index $H\in\left(\frac{1}{2},1\right)$, the three definitions of long-range dependence of Definition (2.3) are equivalent. They hold with the following choice of parameters and slowly varying functions Biagini et al. (2008):

$$\beta=2H-1,\qquad L_1=2H \tag{2.12a}$$

$$\gamma=2-2H\ ,L_2=H(2H-1) \tag{2.12b}$$

$$\delta=2H-1\quad L_3=\pi^{-1}H\Gamma(2H)\sin\pi H \tag{2.12c}$$

Proof. See Biagini et al.(2008), Taqqu (2003) and Doukhan et al.(2003).

Definition 2.4. We say that an $\mathbb{R}^d$- valued random process $X=X_t,\ t\geq 0$, is self-similar or satisfies the property of self-similarity if for every $a>0$, there exists $b>0$ such that,:

$$P(X_{at},\,t\geq 0)=P(bX_t,\,t\geq 0) \tag{2.13}$$

Hence the two processes X_{at} and bX_t have the same finite-dimensional distribution functions, i.e. $\forall t_0,\ldots,t_n\in\mathbb{R}$, and $\forall x_i,\ i,\ldots,n\in\mathbb{R}$.

$$P(X_{at_0}\leq x_0,\ldots,X_{at_n}\leq x_n)=P(bX_{t_0}\leq x_0,\ldots,bX_{t_n}\leq x_n) \tag{2.14}$$

Definition 2.5 If $b=a^{-H}$ in Definition (2.4), then we say that $(X_t)_{t\geq 0}$ is a self-similar process with Hurst index H or that it satisfies the property of (statistical) self-similarity with Hurst index H. See Biagini et al.(2008) and Shiryaev(1999) for more details.

2.4 Wiener Integrals for fBm

Stochastic integrals with respect to fBm can be introduced through their Gaussianity. Stochastic integrals of deterministic functions with respect to a Gaussian process which were introduced in Neveu (1968) are called Wiener integrals. In the case of Brownian motion, they coincided with $It\hat{o}$ integrals. For fBm, they were defined for the first time in the work by Decreusefond and $\ddot{U}stunel$, 1999.

2.5 The Covariance for fBm with $H \in (0,1)$.

Let B_t^H, $t \in [0,T]$, be an fBm with $H \in (0,1)$ on the probability space $\left(\Omega, F^{(H)}, F_t^{(H)}, \mathrm{P}^H\right)$ endowed with the natural filtration $\left(F_t^{(H)}\right)_{t\in[0,T]}$ and the law of the measure, P^H of B_t^H Biagini et al.(2000). If

$$R_H(t,s) := \frac{1}{2}\left(s^{2H} + t^{2H} - \left|t-s\right|^{2H}\right), \quad s,t \ge 0 \tag{2.15}$$

then the covariance $E[B_t^H B_s^H] = R_H(t,s)$. According to Nualart (2003), we obtain the following.

For $H > \frac{1}{2}$, the covariance of the fBm is

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s \left|r-u\right|^{2H-2} du dr \tag{2.16}$$

where $\alpha_H = H(2H-1)$. Also

$$\left|r-u\right|^{2H-2} = \frac{(ru)^{H-\frac{1}{2}}}{\beta\left(2-2H, H-\frac{1}{2}\right)} \cdot \int_0^{r\wedge u} v^{-2H} \left(r-v\right)^{H-\frac{3}{2}} \left(u-v\right)^{H-\frac{3}{2}} dv \tag{2.17}$$

where $\beta(\alpha,\gamma) = \Gamma(\alpha+\gamma)/(\Gamma(\beta)\Gamma(\gamma))$ and the gamma function is given by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \tag{2.18}$$

Let $z = \left(r-v\right)/\left(u-v\right)$.Then

$$\frac{dz}{dv} = \left(r-u\right)/\left(u-v\right)^2 \tag{2.19}$$

But $z(u-v) = r-v \Rightarrow v = (r-zu)/(1-z)$. Also $r-v = r-(r-zu)/(1-z) = z(u-r)/(1-z)$ and $u-v = u-(r-zu)/(1-z) = (u-r)/(1-z)$

$$\therefore \quad dv = \frac{\left(u-v\right)^2}{r-u} dz = \left(\frac{u-r}{1-z}\right)^2 \cdot \frac{dz}{(r-u)} = \left(\frac{r-u}{1-z}\right) dz \tag{2.20}$$

So,

$$\int_0^u v^{1-2H} \left(r-v\right)^{H-\frac{3}{2}} \left(u-v\right)^{H-\frac{3}{2}} dv = \int_{r/u}^\infty \left(\frac{r-zu}{1-z}\right)^{1-2H} \left(\frac{z(u-r)}{1-z}\right)^{H-\frac{3}{2}} \left(\frac{u-r}{1-z}\right)^{H-\frac{3}{2}} \frac{(r-u)}{(1-z)^2} dz$$

$$= \int_{r/u}^\infty \frac{(-1)^{1-2H+H-\frac{3}{2}+H-\frac{3}{2}} (zu-r)^{1-2H} z^{H-\frac{3}{2}} (r-u)^{H-\frac{3}{2}+H-\frac{3}{2}+1}}{(1-z)^{1-2H+H-\frac{3}{2}+H-\frac{3}{2}+2}} dz = (r-u)^{2H-2} \int_{r/u}^\infty (zu-r)^{1-2H} z^{H-\frac{3}{2}} dz$$

(2.21)

Let $x = r/\left(uz\right)$ and suppose $r > u$. $z = \frac{r}{ux}$; $z \to r/u$ as $x \to 1$; $z \to \infty$ as $x \to 0$.

$$dx = \frac{-ru}{(uz)^2} dz = \frac{-r}{u}\left(\frac{ux}{r}\right)^2 dz = \frac{-ux^2}{r} dz \Rightarrow dz = \frac{-r}{ux^2} dx \tag{2.22}$$

Therefore

$$(r-u)^{2H-2}\int_{r/u}^{\infty}(zu-r)^{1-2H}z^{H-\frac{3}{2}}dz = -\left(r-u\right)^{2H-2}\int_{1}^{0}\left[\frac{r}{x}(1-x)\right]^{1-2H}\left(\frac{r}{ux}\right)^{H-\frac{3}{2}}\left(\frac{r}{ux^{2}}\right)dx$$

$$=\left(r-u\right)^{2H-2}\int_{0}^{1}\left(\frac{r}{x}\right)^{1-2H+H-\frac{3}{2}+1}\left(1-x\right)^{1-2H}u^{\frac{3}{2}-H-1}x^{-1}dx \qquad (2.23)$$

$$=\left(r-u\right)^{2H-2}\int_{0}^{1}\left(1-x\right)^{1-2H}r^{\frac{1}{2}-H}u^{\frac{1}{2}-H}x^{H-\frac{3}{2}}dx=(ru)^{\frac{1}{2}-H}\left(r-u\right)^{2H-2}\int_{0}^{1}\left(1-x\right)^{1-2H}x^{H-\frac{3}{2}}dx$$

$$=\beta(2-2H,H-\tfrac{1}{2})(ru)^{\frac{1}{2}-H}(r-u)^{2H-2}$$

Let

$$K_{H}(t,s)=C_{H}s^{\frac{1}{2}-H}\int_{s}^{t}\left(u-s\right)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du \qquad (2.24)$$

where

$$C_{H}=\left[H\left(2H-1\right)/\left(\beta\left(2-2H,H-\tfrac{1}{2}\right)\right)\right]^{\frac{1}{2}}, \qquad t>s \qquad (2.25)$$

Then we have that

$$R_{H}(t,s)=\int_{0}^{t\wedge s}K_{H}(t,u)du \qquad (2.26)$$

By Equation (2.26), it follows that

$$\int_{0}^{t\wedge s}K_{H}(t,u)K_{H}(s,u)du=C_{H}^{2}\int_{0}^{t\wedge s}\left(\int_{u}^{t}\left(y-u\right)^{H-\frac{3}{2}}y^{H-\frac{1}{2}}dy\right)\times\left(\int_{u}^{s}\left(z-u\right)^{H-\frac{3}{2}}z^{H-\frac{1}{2}}dz\right)u^{1-2H}du$$

$$=C_{H}^{2}\int_{0}^{t}\int_{0}^{s}\left(yz\right)^{H-\frac{1}{2}}\left(\int_{0}^{y\wedge s}u^{1-2H}\left(y-u\right)^{H-\frac{3}{2}}\left(z-u\right)^{H-\frac{3}{2}}du\right)dzdy$$

$$=C_{H}^{2}\beta\left(2-2H,H-\tfrac{1}{2}\right)\int_{0}^{t}\int_{0}^{s}\left(y-z\right)^{2H-2}dzdy \qquad (2.27)$$

Putting, $v=y-z,\ dz=-dv,\ v\to y$ as $z\to 0;\ v\to y-s$ as $z\to s,$ we obtain

$$\int_{0}^{t\wedge s}K_{H}(t,u)K_{H}(s,u)du=C_{H}^{2}\beta\left(2-2H,H-\tfrac{1}{2}\right)\int_{0}^{t}\left(-\int_{y}^{y-s}v^{2H-2}dv\right)dy$$

$$=\frac{C_{H}^{2}\beta\left(2-2H,H-\frac{1}{2}\right)}{2H-1}\int_{0}^{t}\left(v^{2H-1}\Big|_{y-s}^{y}\right)dy=\frac{C_{H}^{2}\beta\left(2-2H,H-\frac{1}{2}\right)}{2H-1}\int_{0}^{t}\left(y^{2H-1}-\left(y-s\right)^{2H-1}\right)dy$$

$$=\frac{C_{H}^{2}\beta\left(2-2H,H-\frac{1}{2}\right)}{2H-1}\left[y^{2H}-\left(y-s\right)^{2H}\right]\Big|_{y=0}^{t}$$

$$=\frac{1}{2}\left[t^{2H}+s^{2H}-\left|t-s\right|^{2H}\right]=R_{H}(t,s) \qquad (2.28)$$

III. RESULT: COVARIANCE OF THE FBM OF NASI (1990-2007)

The Hurst parameter for the NASI data of the Nigerian Stock market (NSM) from January 1990 to December 2007 (see Figure 1) was obtained in (Annorzie, 2015, 2018) as: $H=0.78468\approx 0.78$. Using Equation (2.28), the covariance of fBm for the NASI data is given by:

$$R_{0.78468}(t,s)=\frac{1}{2}\left[t^{1.56936}+s^{1.56936}-\left|t-s\right|^{1.56936}\right]$$

$$=E\left[B^{(0.78468)}(t)B^{(0.78468)}\left(s\right)\right]$$

$$=\ 0.114.$$

IV. CONCLUSION

This result illustrates an example of the covariance of fBm of a stable, persistent or emerging stock market where $H\in(0.5,1)$. The Fractional Brownian motion, as a stochastic process B_{t}^{H}, has one parameter H, $H\in(0,1)$, which can be characterized by its self-similarity nature. If we consider a scale parameter

$a > 0$, the process $\left(B^H_{t+a\tau} - B^H_\tau\right)$ has the same moments as $\left(a^H B^H_\tau\right)$. It is a non-stationary process, that is, its covariance function is a function of time, t, with $E\left[B^H_t B^H_s\right]$, where $t > s$.

fBm has been of interest in graphics because as a first approximation, it is a useful model of terrain. The main advantage of fBm as a model of terrain is a remarkable compactness of representation. Depending on how much deterministic data is included, the data base can be from two numbers to a few hundreds, to represent terrain that ultimately contains thousands or million of polygons. The second big advantage, due to its self-similarity nature, is that unlimited amount of details can be generated. Alternatively, the disadvantages include the fact that to generate a surface, pure recursive subdivision is not sufficient which will also complicate the subdivision algorithm. Also, it has limited flexibility with basically only one parameter to be adjusted to generate different terrains.

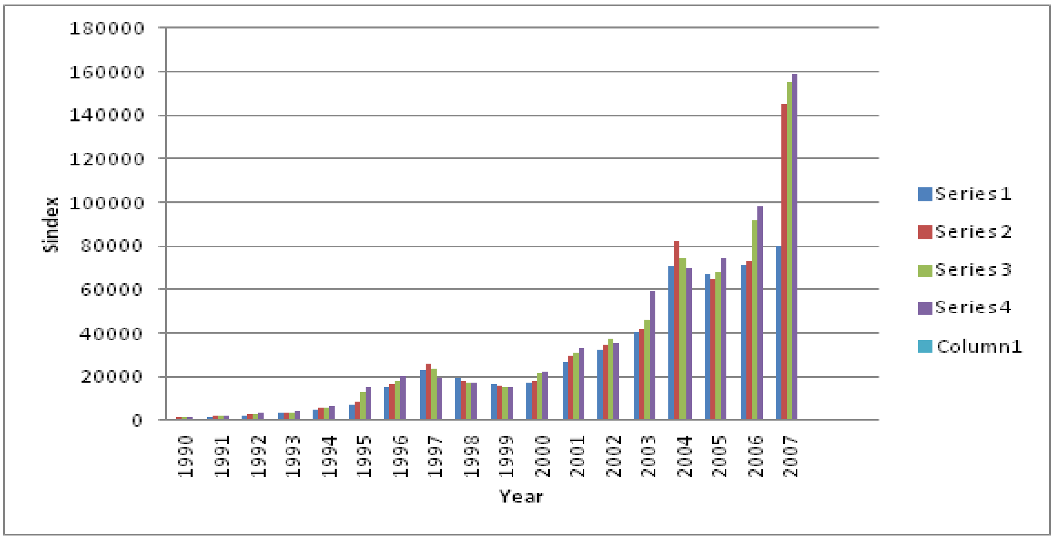

Figure 1: The Empirical plot of the NASI (1990-2007)

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