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# Initial and Boundary Value Problems Involving the Inhomogeneous Weber Equation and the Nield-Kuznetsov Parametric Functions

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**ABSTRACT**

*Initial and boundary value problems of the inhomogeneous Weber differential equation are treated in this work. General solutions are expressed in terms of the parametric Nield-Kuznetsov functions of the first and second kinds, and are computed when the forcing function is a constant or a variable function of the independent variable.*

***Keywords*** *Weber inhomogeneous equation, parametric Nield-Kuznetsov functions.*

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## I. INTRODUCTION

In a recent article, Alzahrani *et al*. [1] discussed how Weber's differential equation can be used in modelling flow through porous layers with variable permeability. In particular, they showed that the inhomogeneous Weber ordinary differential equations (ODE) of the form

$$u'' + (\frac{y^2}{4} - a)u = f(y) \;\; ; a \in \Re \qquad \ldots(1)$$

wherein prime notation denotes ordinary differentiation with respect to the independent variable *y*, plays an important role in the study of transition layer introduced by Nield and Kuznetsov, [2].

Equation (1) and its more general forms have received considerable attention in the literature due to its many applications in mathematical physics, (*cf*. [3] and the references therein). The elegant discussion of equation (1) by Temme, [3], illustrates that the homogeneous part of (1) possesses the parabolic cylindrical functions, $W(a, \mp y)$, as solutions, where $y$ and $a$ are real numbers. These solutions represent a linearly independent, numerically satisfactory pair of solutions for all $y \in \Re$, with a Wronskian given by $\mathfrak{W}(W(a,x), W(a,-x))=1$. Computational techniques of the functions $W(a, \mp y)$ have been provided by many authors, and include the efficient algorithms developed by Temme, [4], and Gil *et al.*, [5].

Computations of solution to (1) when the forcing function $f(y)$ is a constant function of the independent variable have been obtained by Abu Zaytoon et al. [6], who used computational procedures outlined in Gil *et al.*, [5] to obtain solutions to initial and boundary value problems involving (1) with a constant forcing function *f(y)*.

In the work of Alzahrani *et al*. [1], general solutions to the inhomogeneous equation (1) were expressed in terms of the parametric Nield-Kuznetsov integral functions of the first and second kinds, valid for constant and variable forcing functions, respectively. Series expressions for the said Nield-Kuznetsov functions were also developed.

In the current work, we provide solutions to initial and boundary value problems involving equation (1) and subject to the initial conditions:

$$u(0) = \alpha \text{ and } u'(0) = \beta \qquad \ldots(2)$$

or subject to the boundary conditions:

$u(a_1) = b_1$ and $u(a_2) = b_2$ …(3)

where $\alpha$ , $\beta$ , $a_1$, $a_2$, $b_1$ and $b_2$ are real numbers.

Solutions and computations are provided for the cases of $f(y)$ being either a constant function or a variable function of the independent variable, in order to study the effects of the forcing function on the behavior of the solutions obtained.

## II. SOLUTION PROCEDURE

General solution to equation (1) is given by, [1], [6]:

$$u = c_{1w} W(a, y) + c_{2w} W(a, -y) - K_w(a, y) \quad \text{…(4)}$$

where $c_{1w}$, $c_{2w}$ are arbitrary constants, $W(a, y)$ and $W(a, -y)$ are the Weber functions with parameter *a*, [3], and $K_w(a, y)$ is the parametric Nield-Kuznetsov function of the second-kind, defined by, [1]

$$K_w(a, y) = W(a, -y)\int_0^y F(t) W'(a, t)\,dt + W(a, y)\int_0^y F(t) W'(a, -t)\,dt \quad \text{…(5)}$$

with first derivative given by

$$K'_w(a, y) = W'(a, y)\int_0^y F(t) W'(a, -t)\,dt - W'(a, -y)\int_0^y F(t) W'(a, t)\,dt - F(y) \quad \text{…(6)}$$

wherein $F'(y) = f(y)$.

When the forcing function is a constant, say $f(y) = \kappa$ , general solution (4) takes the form

$$u = c_1 W(a, y) + c_2 W(a, -y) - \kappa N_w(a, y) \quad \text{…(7)}$$

where $N_w(a, y)$ is the parametric Nield-Kuznetsov function of the first-kind, defined by, [1]:

$$N_w(a, y) = W(a, y)\int_0^y W(a, -t)\,dt - W(a, -y)\int_0^y W(a, t)\,dt \quad \text{…(8)}$$

with first derivatives given by

$$N'_w(a, y) = W'(a, y)\int_0^y W(a, -t)\,dt + W'(a, -y)\int_0^y W(a, t)\,dt \,. \quad \text{…(9)}$$

In order to obtain solutions to the initial and boundary value problems, general solutions (4) and (7) must satisfy condition (2) for initial value problem, and condition (3) for boundary value problem. This requires determination of the arbitrary constants appearing in (4) and (7). In what follows, the arbitrary constants are determined for cases of constant and variable forcing functions.

In arriving at the expressions for the arbitrary constants, use is made of the following values at zero of the Nield-Kuznetsov functions of the first and second kinds, obtained directly from equations (5), (6), (8) and (9), namely:

$$N_w(a, 0) = N'_w(a, 0) = 0 \quad \text{…(10)}$$

$$K_w(a, 0) = 0 \;;\; K'_w(a, 0) = -F(0) \quad \text{…(11)}$$

and values of the Weber functions $W(a, 0)$ and $W'(a, 0)$, reported in the literature [3,4,5], as:

$$W(a, 0) = \frac{1}{(2)^{3/4}}\left|\frac{\Gamma(\frac{1}{4} + \frac{ia}{2})}{\Gamma(\frac{3}{4} + \frac{ia}{2})}\right|^{1/2} \quad \text{…(12)}$$

$$W'(a,0) = -\frac{1}{(2)^{1/4}}\left|\frac{\Gamma(\frac{3}{4}+\frac{ia}{2})}{\Gamma(\frac{1}{4}+\frac{ia}{2})}\right|^{1/2} \quad \ldots(13)$$

$$W(0,0) = \frac{1}{(2)^{3/4}}\left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)^{1/2} \quad \ldots(14)$$

$$W'(0,0) = -\frac{1}{(2)^{1/4}}\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\right)^{1/2}. \quad \ldots(15)$$

## III. SOLUTION TO THE INITIAL VALUE PROBLEM (IVP)

**III.1. Constant Forcing Function:**

When $f(y) = \kappa$, where $\kappa$ is a given constant, the use of initial conditions (2) in the general solution (7) yields the following values for the arbitrary constants $c_{1w}$, $c_{2w}$ :

$$c_{1w} = \frac{\alpha W'(a,0) + \beta W(a,0)}{2W(a,0)W'(a,0)} \quad \ldots(16)$$

$$c_{2w} = \frac{\alpha W'(a,0) - \beta W(a,0)}{2W(a,0)W'(a,0)}. \quad \ldots(17)$$

**III.2. Variable Forcing Function:**

When $f(y)$ is a variable function of *y*, using initial conditions (2) in gerenal solution (4) yields the following values for the arbitrary constants $c_{1w}$, $c_{2w}$ :

$$c_{1w} = \frac{\alpha W'(a,0) + W(a,0)\left(\beta + K'_w(a,0)\right)}{2W(a,0)W'(a,0)} \quad \ldots(18)$$

$$c_{2w} = \frac{\alpha W'(a,0) - W(a,0)\left(\beta + K'_w(a,0)\right)}{2W(a,0)W'(a,0)}. \quad \ldots(19)$$

## IV. SOLUTION TO THE BOUNDARY VALUE PROBLEM (BVP)

**IV.1. Constant Forcing Function:**

When $f(y) = \kappa$, using boundary conditions (3) in gerenal solution (7) yields the following values for the arbitrary constants $c_{1w}$ and $c_{2w}$ :

$$c_{1w} = \frac{b_1 W(a,-a_2) - b_2 W(a,-a_1) + \kappa\left[N_w(a,a_1)W(a,-a_2) - N_w(a,a_2)W(a,-a_1)\right]}{W(a,a_1)W(a,-a_2) - W(a,a_2)W(a,-a_1)} \quad \ldots(20)$$

$$c_{2w} = \frac{b_1 W(a,a_2) - b_2 W(a,a_1) + \kappa\left[N_w(a,a_1)W(a,a_2) - N_w(a,a_2)W(a,a_1)\right]}{W(a,-a_1)W(a,a_2) - W(a,-a_2)W(a,a_1)}. \quad \ldots(21)$$

**IV.2. Variable Forcing Function:**

When $f(y)$ is a variable function of *y*, using boundary conditions (3) in general solution (4) yields the following values for the arbitrary constants $c_{1w}$ and $c_{2w}$ :

$$c_{1w} = \frac{b_1 W(a,-a_2) - b_2 W(a,-a_1) + K_w(a,a_1) W(a,-a_2) - K_w(a,a_2) W(a,-a_1)}{W(a,a_1) W(a,-a_2) - W(a,a_2) W(a,-a_1)} \quad \ldots(22)$$

$$c_{2w} = \frac{b_1 W(a,a_2) - b_2 W(a,a_1) + K_w(a,a_1) W(a,a_2) - K_w(a,a_2) W(a,a_1)}{W(a,-a_1) W(a,a_2) - W(a,-a_2) W(a,a_1)}\,. \quad \ldots(23)$$

## V. COMPUTATIONS OF SOLUTIONS TO IVP AND BVP

### V.1. Series Expressions for the Parametric Nield-Kuznetsov Functions

In order to determine values of the arbitrary constants, given by expressions (16)-(23), and to evaluate solutions to IVP and BVP, we need to evaluate Weber functions and the parametric Nield-Kuznetsov functions at the given values of the independent variable, *y*. These have been developed in [1] and in [6], as follows.

The following series expressions have been developed in [3,4,5], for the Weber functions $W(a,y)$ and $W(a,-y)$:

$$W(a,y) = W(a,0)\sum_{n=0}^{\infty} \rho_n(a)\frac{y^{2n}}{(2n)!} + W'(a,0)\sum_{n=0}^{\infty} \delta_n(a)\frac{y^{2n+1}}{(2n+1)!} \quad \ldots(24)$$

$$W(a,-y) = W(a,0)\sum_{n=0}^{\infty} \rho_n(a)\frac{y^{2n}}{(2n)!} - W'(a,0)\sum_{n=0}^{\infty} \delta_n(a)\frac{y^{2n+1}}{(2n+1)!} \quad \ldots(25)$$

$$W'(a,y) = W(a,0)\sum_{n=1}^{\infty} \rho_n(a)\frac{y^{2n-1}}{(2n-1)!} + W'(a,0)\sum_{n=1}^{\infty} \delta_n(a)\frac{y^{2n}}{(2n)!} \quad \ldots(26)$$

$$W'(a,-y) = W'(a,0)\sum_{n=1}^{\infty} \delta_n(a)\frac{y^{2n}}{(2n)!} - W(a,0)\sum_{n=1}^{\infty} \rho_n(a)\frac{y^{2n-1}}{(2n-1)!} \quad \ldots(27)$$

$$\rho_{n+2} = a\rho_{n+1} - \frac{1}{2}(n+1)(2n+1)\rho_n \quad \ldots(28)$$

$$\delta_{n+2}(a) = a\delta_{n+1}(a) - \frac{1}{2}(n+1)(2n+3)\delta_n(a) \quad \ldots(29)$$

$$\rho_0(a) = 1\,;\ \rho_1(a) = a\,;\ \delta_0(a) = 1\,;\ \delta_1(a) = a\,. \quad \ldots(30)$$

Using equations (24)-(30) in the definitions of $N_w(a,y)$ and $K_w(a,y)$, given by (5) and (8), we obtain the following series expressions:

$$N_w(a,y) = 2W(a,0)W'(a,0)\begin{bmatrix} \left\{\sum_{n=0}^{\infty} \delta_n(a)\frac{y^{2n+1}}{(2n+1)!}\right\}\left\{\sum_{n=0}^{\infty} \rho_n(a)\frac{y^{2n+1}}{(2n+1)!}\right\} \\ -\left\{\sum_{n=0}^{\infty} \rho_n(a)\frac{y^{2n}}{(2n)!}\right\}\left\{\sum_{n=0}^{\infty} \delta_n(a)\frac{y^{2n+2}}{(2n+2)!}\right\} \end{bmatrix} \quad \ldots(31)$$

$$K_w(a,y) = 2W'(a,0)W(a,0)[\sum_{n=0}^{\infty} \rho_n(a)\frac{y^{2n}}{(2n)!}]\int_0^y [\sum_{n=1}^{\infty} \delta_n(a)F(t)\frac{t^{2n}}{(2n)!}]dt\,. \quad \ldots(32)$$

### V.2. Numerical Experiment

Assume that it is required to solve (1) subject to initial conditions $u(0) = 2, u'(0) = 1$ or boundary conditions $u(0) = 1, u(1) = 2$. The forcing function is either constant, $f(y) = \kappa = \frac{1}{\pi}$ and $f(y) = \kappa = -\frac{1}{\pi}$, or variable $f(y) = y$, $f(y) = y^2$ and $f(y) = \sin y$. Then, expressions (16)-(23) are evaluated for the arbitrary constants using expressions (24)-(32). Solutions to IVP, and to BVP, are then evaluated and plotted over the interval $0 \le y \le 1$ for the various values of parameter *a* = 0, 1, 2, -1, as discussed in the next section.

**V.3. Solution to the Initial Value Problem (IVP)**

Values of the arbitrary constants, computed using (16)-(19) are given in **Table 1** for the various values of parameter *a* tested. These values are independent of the forcing function, and they have the same values for both constant and variable forcing function in equation (1).

**Table 1.** Values of arbitrary constants in IVP with constant forcing function for different values of parameter $a$ .

| All $f(y)$ | $c_{1w}$ | $c_{2w}$ |
|---|---|---|
| $a = 0$ | $-0.04502460$ 512 | 2.00050673 9 |
| $a = 1$ | 0.635608249 0 | 2.09857042 9 |
| $a = 2$ | 1.06562955 2 | 2.26617879 8 |
| $a = -1$ | 0.635608249 0 | 2.09857042 9 |

Solution curves for $f(y) = \kappa = \mp 1/\pi$ are shown in **Figure 1(a-f). Figure 1(a-d)** provide a comparison between the solution profiles of $f(y) = 1/\pi$ and $f(y) = -1/\pi$ , for each of the values of parameter *a* tested, and show the relative positions of the solution curves. For positive values of *a*, the solution curves corresponding to both forcing functions are exponential, and get closer to each other as *a* increases. This might point to the conclusion that when *a* gets large enough one might expect the solution to (1), with constant forcing function, to become independent of the forcing function.

When *a* = 0, **Figure 1(a)** shows an almost linear solution curve for $f(y) = 1/\pi$ , and a parabolic curve for $f(y) = -1/\pi$ whose vertex does not fall within the interval $0 \le y \le 1$ . **Figure 1(d)** shows parabolic curves, when *a* = -1, with vertices representing maximum points over the interval $0 \le y \le 1$ .

Comparison of the solution curves for different values of *a* is given in **Figure 1(e)**, for $f(y) = 1/\pi$ , and in **Figure 1(f)**, for $f(y) = -1/\pi$ . The solution curve corresponding to *a* = 0 is an almost linear, increasing curve for both constant forcing functions tested. Above this curve lie the exponentially increasing solution curves corresponding to *a > 0*, and below which lies the parabolic curve that corresponds to *a < 0*.

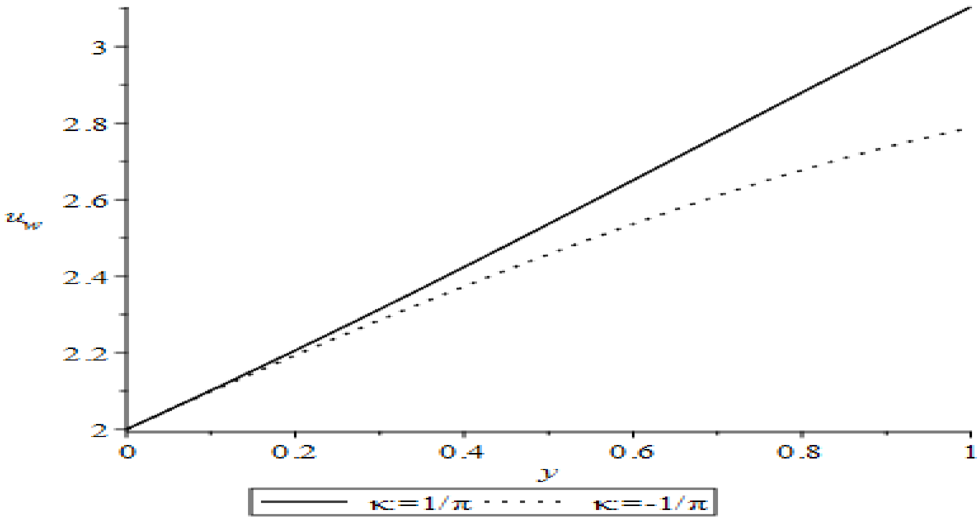


**Fig. 1(a).** Solution to IVP for Constant Forcing Function $f(y) = \kappa = \mp 1/\pi$ ; *a=0.*

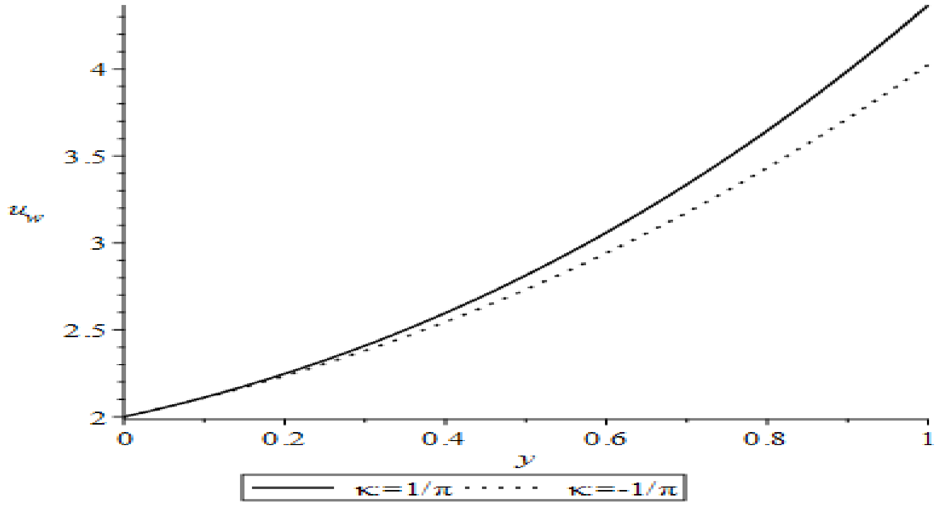


**Fig. 1(b).** Solution to IVP for Constant Forcing Function $f(y) = \kappa = \mp 1/\pi$ ; *a=1.*

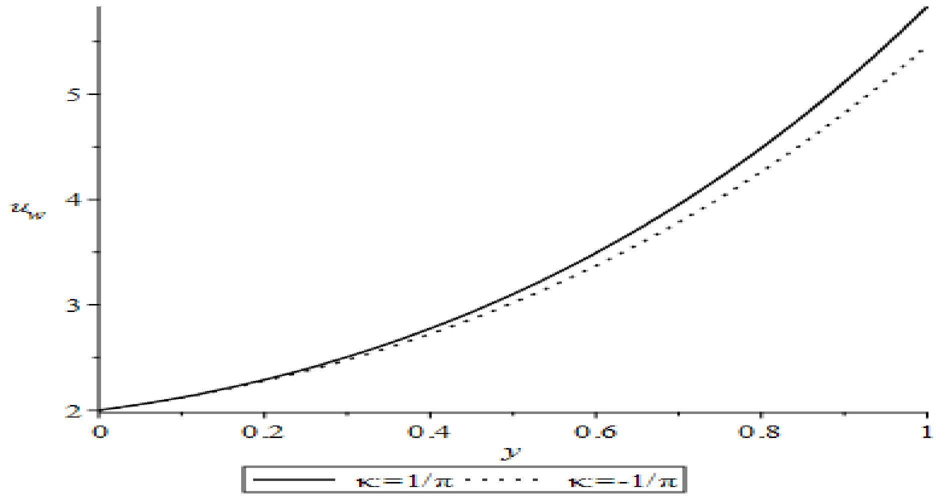


**Fig. 1(c).** Solution to IVP for Constant Forcing Function $f(y) = \kappa = \mp 1/\pi$ ; *a=2.*

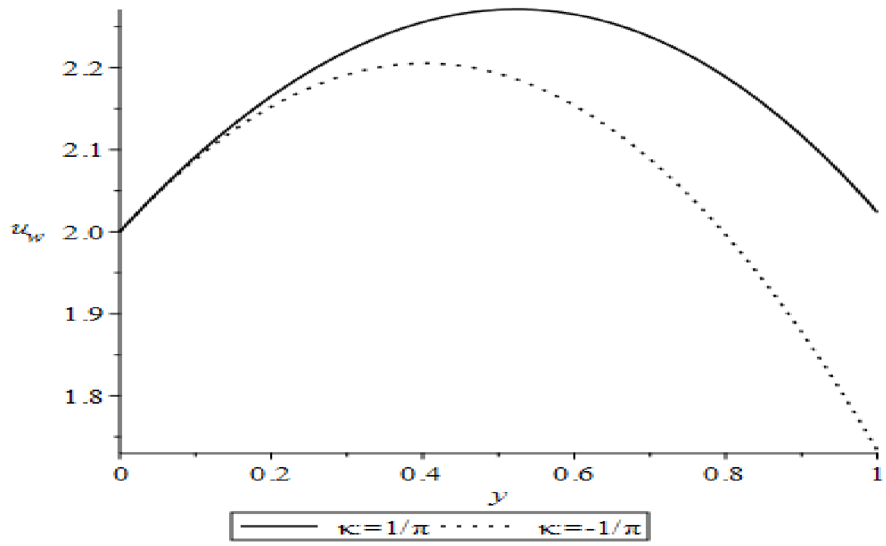


**Fig. 1(d).** Solution to IVP for Constant Forcing Function $f(y) = \kappa = \mp 1/\pi$ ; *a=-1.*

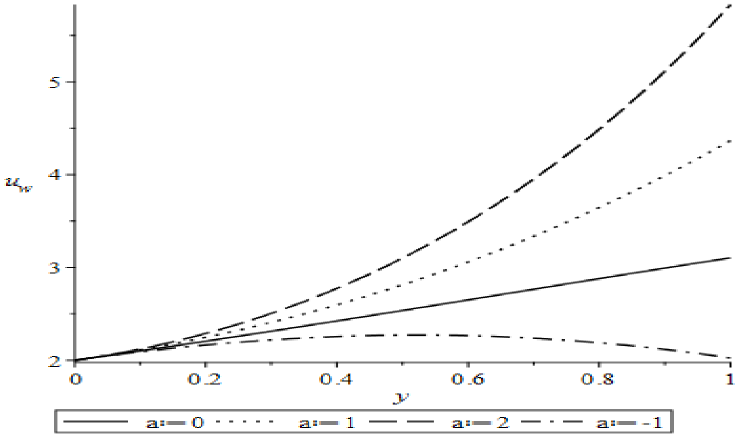


**Fig. 1(e).** Solution to IVP for Constant Forcing Function $f(y) = \kappa = 1/\pi$ and Different Values of *a*.

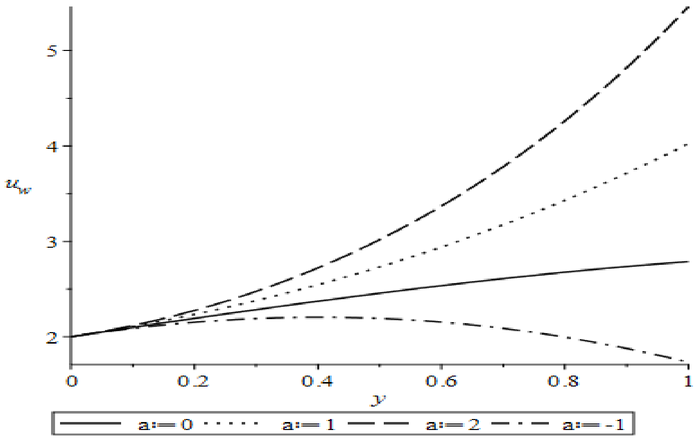


**Fig. 1(f).** Solution to IVP for Constant Forcing Function $f(y) = \kappa = -1/\pi$ and Different Values of *a*.

Solution curves for the IVP with variable forcing function are shown in **Figures 2(a-g).** Relative positions of the curves for each value of *a* are illustrated in **Figures 2(a-d)**, and their shapes depend on the value of *a* in much the same way as for the case of constant forcing function.

Effects of changing *a* for each of the variable forcing functions $f(y) = y$, $f(y) = y^2$ and $f(y) = \sin y$, are illustrated in **Figures 2(e,f,g)**, respectively. Again, the solution curve corresponding to *a* = 0 is an almost linear, increasing curve for all variable forcing functions tested. Above this curve lie the exponentially increasing solution curves corresponding to *a > 0*, and below which lies the parabolic curve that corresponds to *a < 0*. This pattern persists for all forcing functions tested.

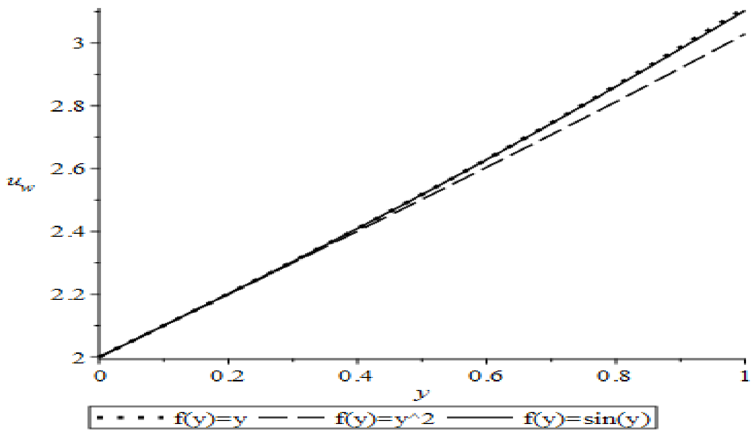


**Fig. 2(a).** Solution to IVP for Variable Forcing Function $f(y)$ ; *a=0*

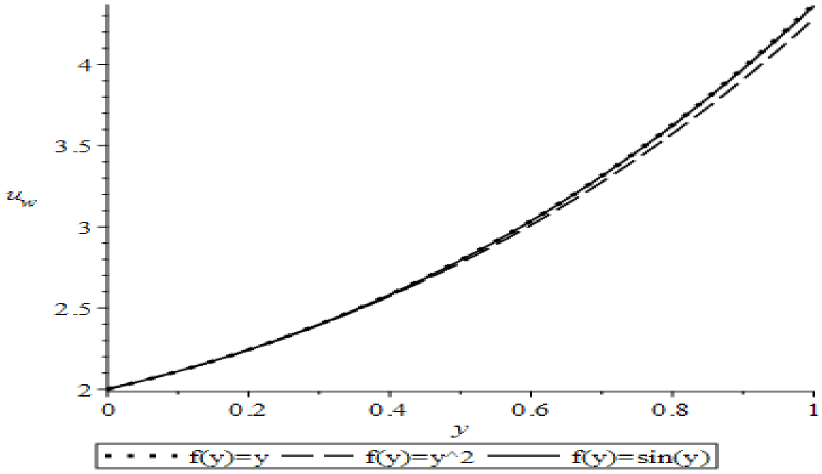


**Fig. 2(b).** Solution to IVP for Variable Forcing Function $f(y)$ ; *a=1.*

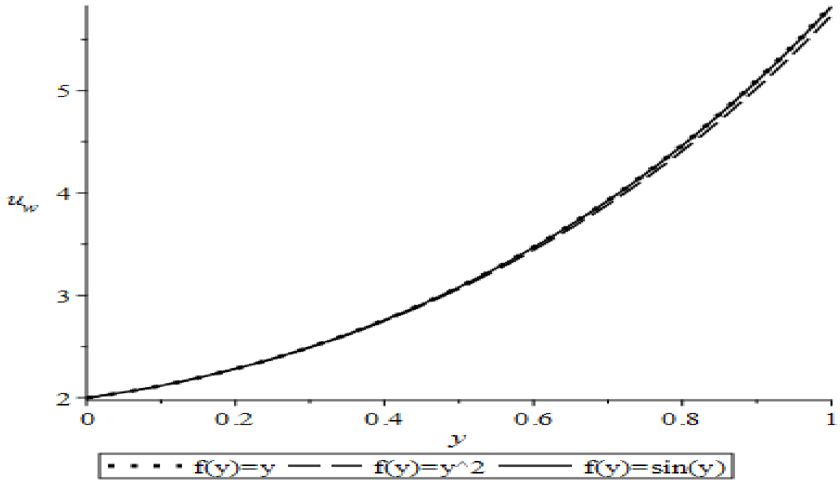


**Fig. 2(c).** Solution to IVP for Variable Forcing Function $f(y)$ ; *a=2.*

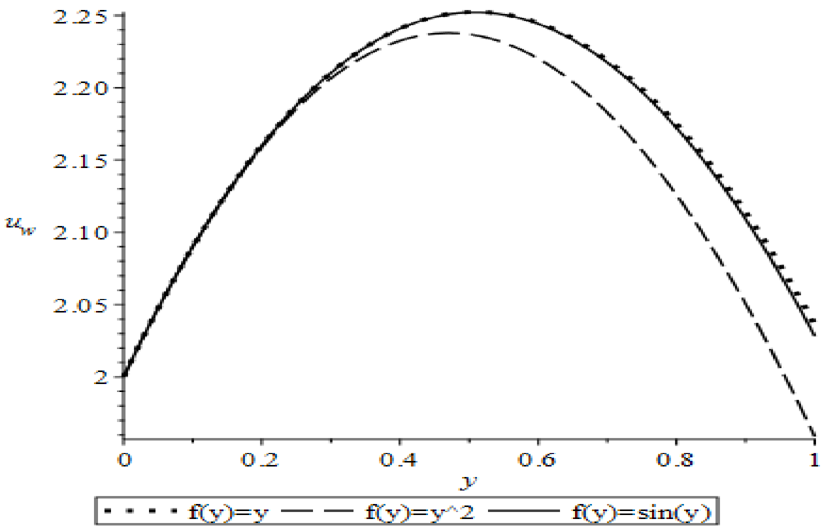


**Fig. 2(d).** Solution to IVP for Variable Forcing Function $f(y)$ ; *a=-1.*

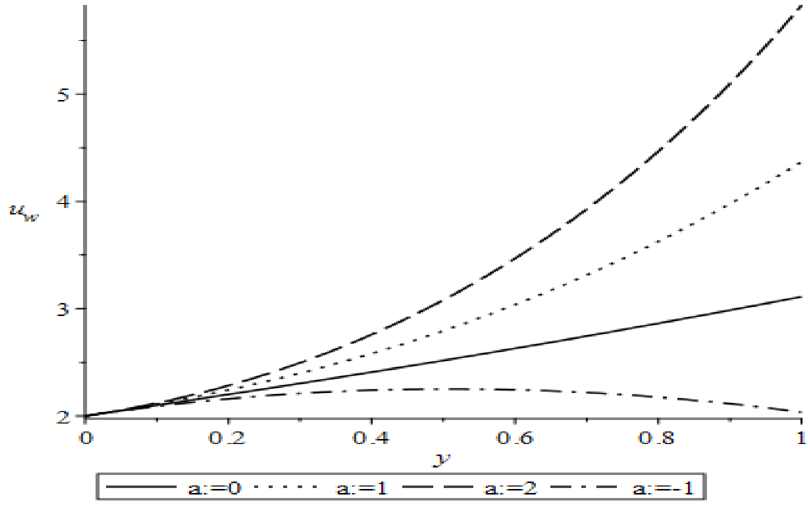


**Fig. 2(e).** Solution to IVP for $f(y) = y$ and Different Values of *a*.

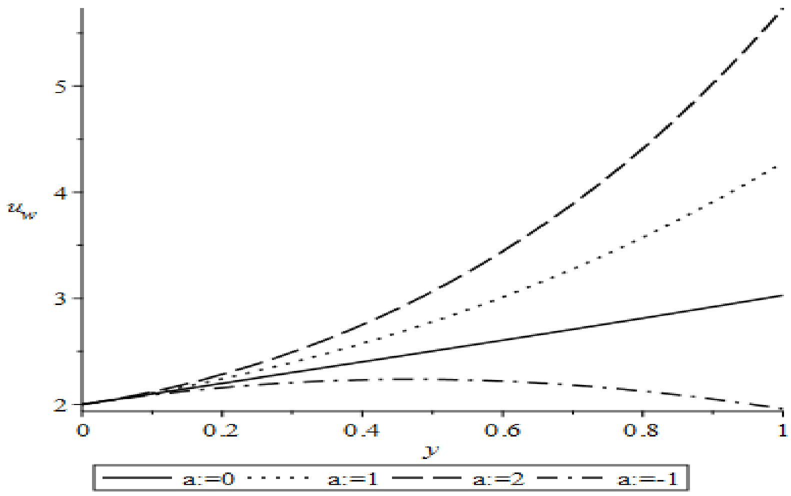


**Fig. 2(f).** Solution to IVP for $f(y) = y^2$ and Different Values of *a*.

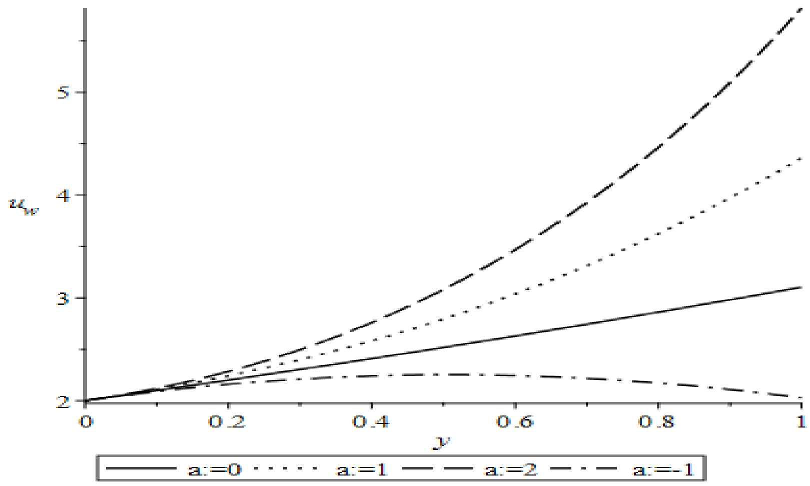


**Fig. 2(g).** Solution to IVP for $f(y) = \sin y$ and Different Values of *a*.

**V.4. Solution to the Boundary Value Problem (BVP)**

Values of the arbitrary constants, computed using (20) and (21), are given in **Table 2** for the various values of parameter *a* and constant forcing functions.

**Table 2.** Values of arbitrary constants in BVP with constant forcing function for different values of parameter $a$ .

| | $f(y) = \kappa = \frac{1}{\pi}$ | $f(y) = \kappa = -\frac{1}{\pi}$ |
|---|---|---|
| $a = 0$ | $c_{1w} = -0.404814662\ 7$<br>$c_{2w} = 1.38255573\ 1$ | $c_{1w} = -0.731738018\ 6$<br>$c_{2w} = 1.70947908\ 6$ |
| $a = 1$ | $c_{1w} = 0.487465431\ 4$<br>$c_{2w} = 0.879623908\ 5$ | $c_{1w} = 0.271486559\ 2$<br>$c_{2w} = 1.09560278\ 1$ |
| $a = 2$ | $c_{1w} = 0.980515307\ 8$<br>$c_{2w} = 0.685388865\ 2$ | $c_{1w} = 0.815467277\ 2$<br>$c_{2w} = 0.850436896\ 0$ |
| $a = -1$ | $c_{1w} = -0.488908752\ 8$<br>$c_{2w} = 1.85599809\ 1$ | $c_{1w} = -0.744556596\ 9$<br>$c_{2w} = 2.11164593\ 5$ |

Solution curves for $f(y) = \kappa = \mp 1/\pi$ are shown in **Figures 3(a-f). Figures 1(a-d)** provide a comparison of the solution profiles of $f(y) = 1/\pi$ and $f(y) = -1/\pi$ for each of the values of parameter *a* tested, and show the relative positions of the solution curves. For positive values of *a*, the solution curves corresponding to both forcing functions are parabolic with vertices representing minimum points. For negative *a*, solution curves are parabolic with vertices representing maximum points. When *a* = 0, the solution curve corresponding to $f(y) = -1/\pi$ is parabolic with a vertex representing a maximum point while the solution curve corresponding to $f(y) = 1/\pi$ is an increasing, almost linear curve. In all cases, the curves corresponding to $f(y) = -1/\pi$ are higher than the curves corresponding to $f(y) = 1/\pi$ .

Comparison of the solution curves for different values of *a* is given in **Figure 3(e)**, for $f(y) = 1/\pi$ , and in **Figure 3(f)**, for $f(y) = -1/\pi$ . The solution curve corresponding to *a* = 0 is an almost linear, increasing curve for both constant forcing functions tested. Above this curve lie the parabolic solution curves corresponding to *a* < *0*, and below which lies the parabolic curve that corresponds to *a* > *0*.

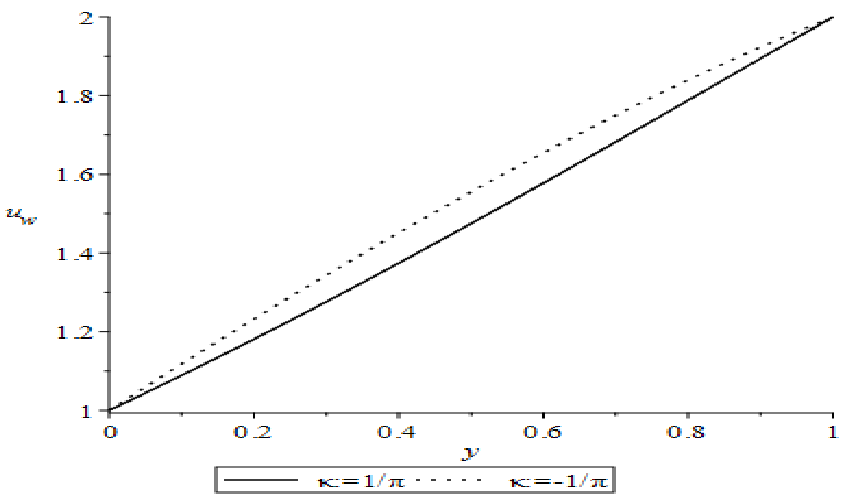


**Fig. 3(a).** Solution to BVP for Constant Forcing Function $f(y) = \kappa = \mp 1/\pi$ ; *a=0.*

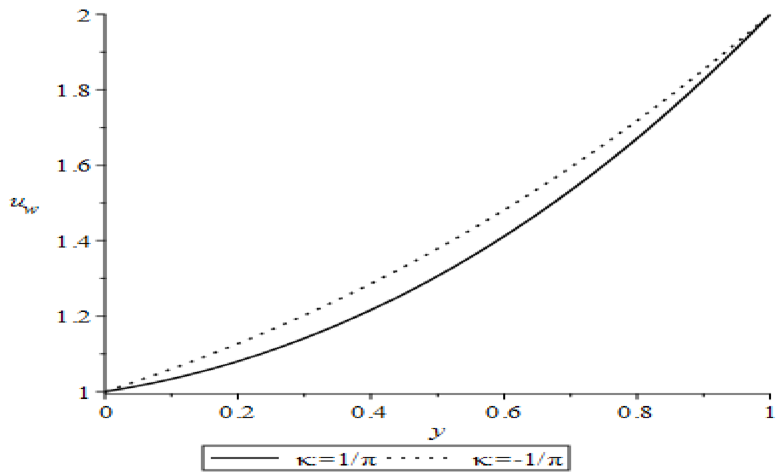


**Fig. 3(b).** Solution to BVP for Constant Forcing Function $f(y) = \kappa = \mp 1/\pi$ ; *a=1.*

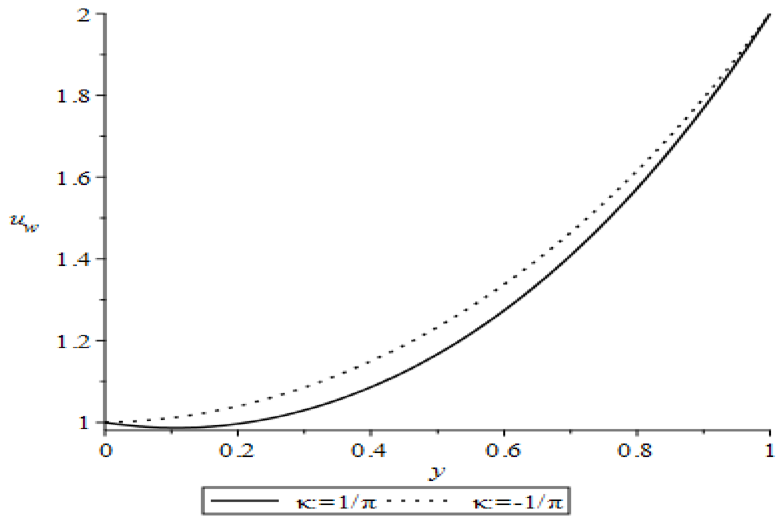


**Fig. 3(c).** Solution to BVP for Constant Forcing Function $f(y) = \kappa = \mp 1/\pi$ ; *a=2.*

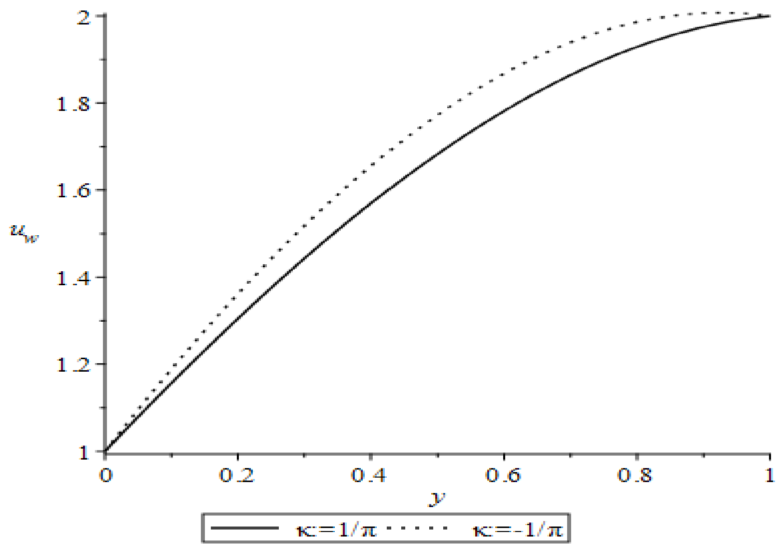


**Fig. 3(d).** Solution to BVP for Constant Forcing Function $f(y) = \kappa = \mp 1/\pi$ ; *a=-1.*

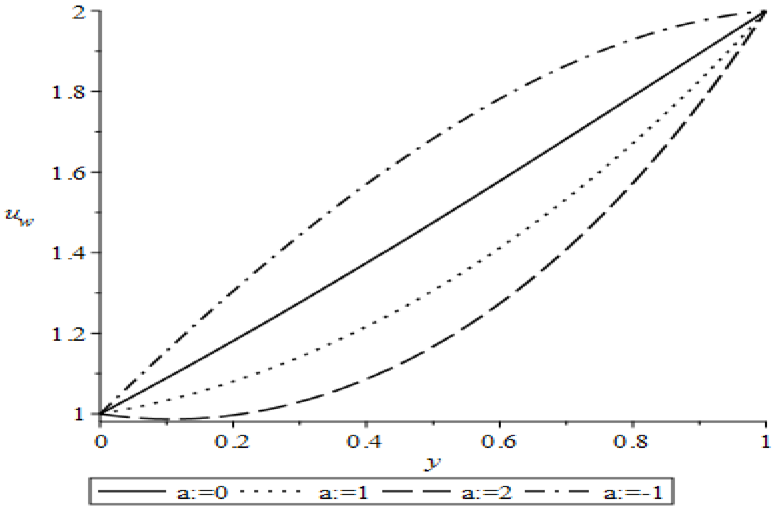


**Fig. 3(e).** Solution to BVP for Constant Forcing Function $f(y) = \kappa = 1/\pi$ and Different Values of *a*.

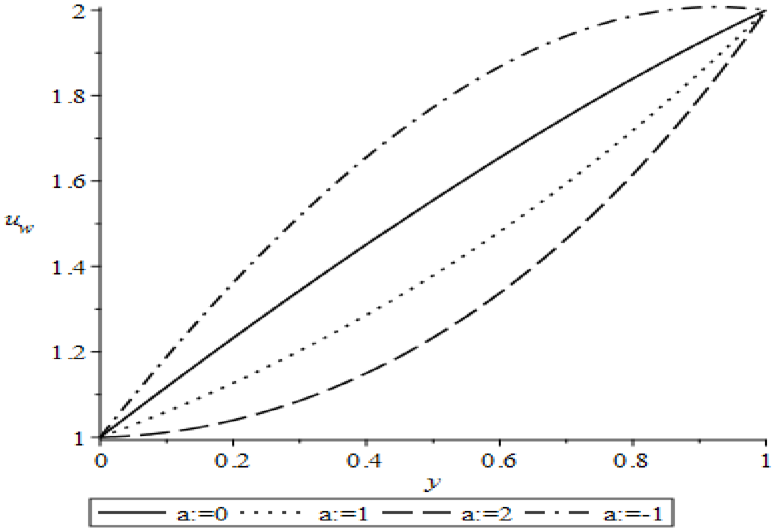


**Fig. 3(f).** Solution to BVP for Constant Forcing Function $f(y) = \kappa = -1/\pi$ and Different Values of *a*.

Values of the arbitrary constants, computed using (22) and (23), are given in **Table 3** for the various values of parameter *a* and variable forcing functions.

**Table 3.** Values of arbitrary constants in BVP with variable forcing function for different values of parameter *a* .

| | $f(y) = y$ | $f(y) = y^2$ | $f(y) = \sin\ y$ |
|---|---|---|---|
| $a = 0$ | $c_{1w} = -0.396690350$ 7<br>$c_{2w} = 1.37443141$ 9 | $c_{1w} = -0.482355352$ 0<br>$c_{2w} = 1.46009642$ 0 | $c_{1w} = -0.404848146$ 7<br>$c_{2w} = 1.38258921$ 5 |
| $a = 1$ | $c_{1w} = 0.489170650$ 9<br>$c_{2w} = 0.877918688$ 9 | $c_{1w} = 0.433500765$ 9<br>$c_{2w} = 0.933588574$ 3 | $c_{1w} = 0.483774402$ 8<br>$c_{2w} = 0.883314937$ 5 |
| $a = 2$ | $c_{1w} = 0.979208378$ 4<br>$c_{2w} = 0.686695794$ 6 | $c_{1w} = 0.937340165$ 7<br>$c_{2w} = 0.728564007$ 8 | $c_{1w} = 0.975627812$ 7<br>$c_{2w} = 0.690276360$ 6 |
| $a = -1$ | $c_{1w} = -0.477898756$ 1<br>$c_{2w} = 1.84498809$ 5 | $c_{1w} = -0.546034310$ 5<br>$c_{2w} = 1.91312364$ 9 | $c_{1w} = -0.484930278$ 8<br>$c_{2w} = 1.85201961$ 7 |

Solution curves for the BVP with variable forcing function are shown in **Figures 4(a-g).** Relative positions of the curves for each value of *a* are illustrated in **Figures 4(a-d)**, and their shapes depend on the value of *a* in much the same way as for the case of constant forcing function. For the three variable forcing functions tested for $0 \le y \le 1$, solution curves are very close to each other and assume similar parabolic shapes. This might be indicative of lesser influence the forcing functions have on the solution, as compared to the influence of parameter *a*, as illustrated in **Figures 4(e,f,g)**, for each of the variable forcing functions $f(y) = y$, $f(y) = y^2$ and $f(y) = \sin y$. For each of the variable forcing functions, the solution curve corresponding to $a = 0$ is almost linear, increasing curve for all variable forcing functions tested. Above this curve lies the parabolic solution curve corresponding to $a < 0$, and below which lie the parabolic curve that corresponds to $a > 0$. This pattern persists for all forcing functions tested.

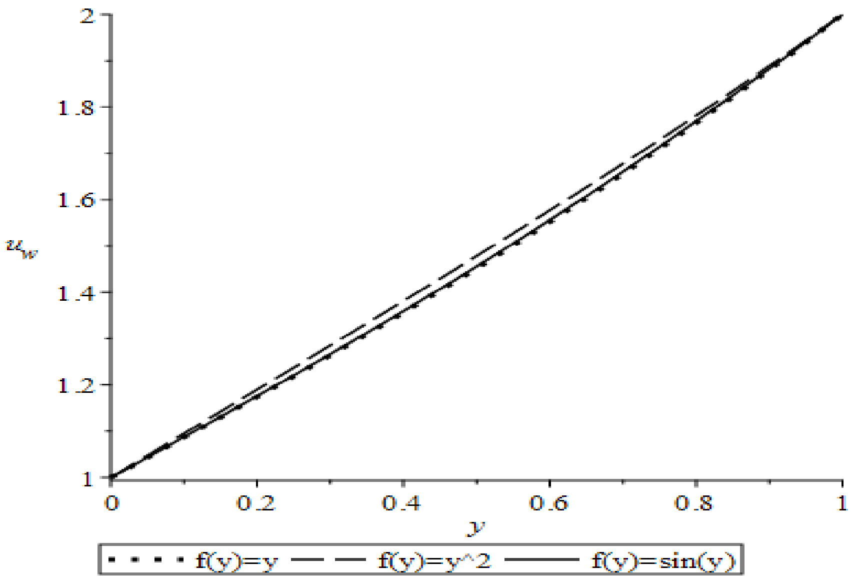


**Fig. 4(a).** Solution to BVP for Variable Forcing Function $f(y)$; *a=0.*

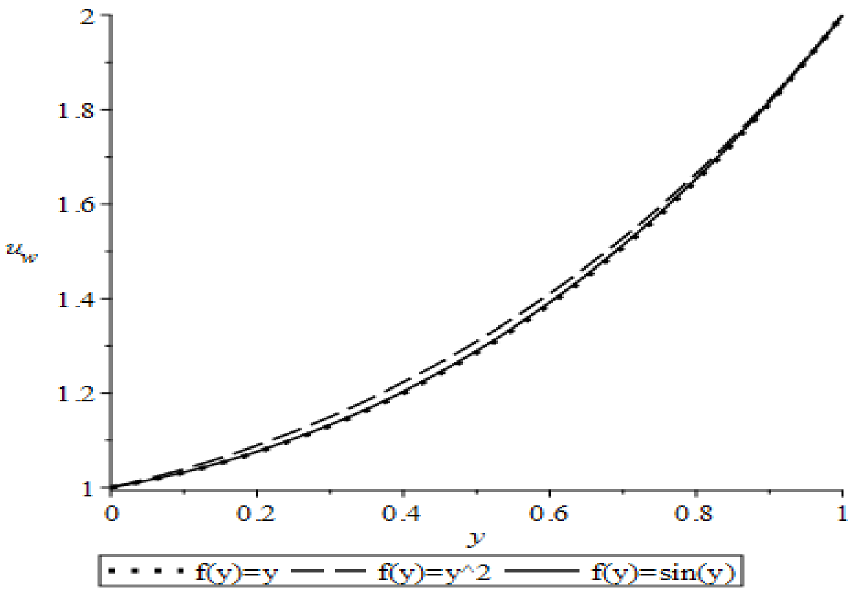


**Fig. 4(b).** Solution to BVP for Variable Forcing Function $f(y)$; *a=1.*

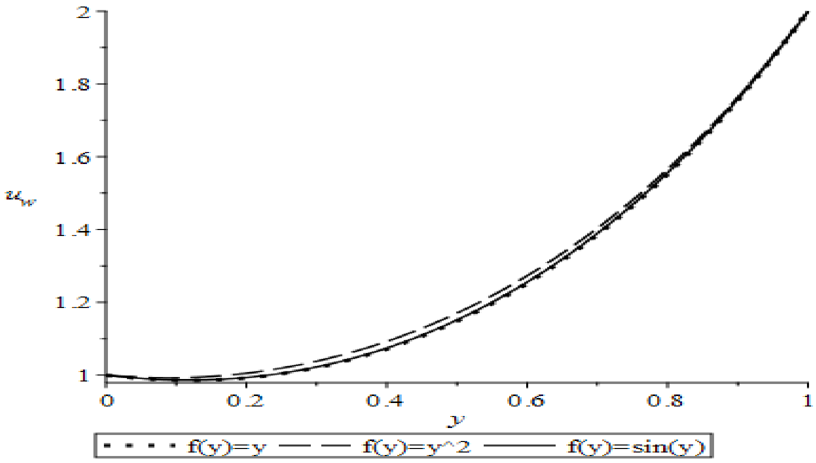


**Fig. 4(c).** Solution to BVP for Variable Forcing Function $f(y)$ ; *a=2.*

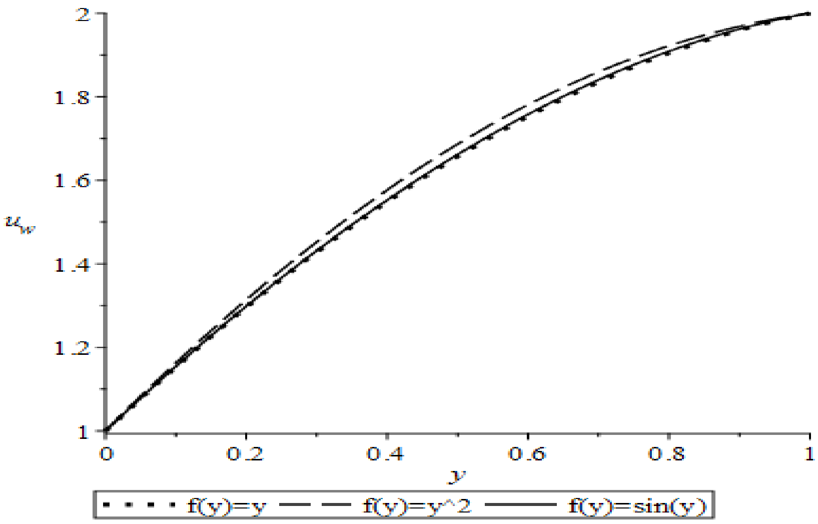


**Fig. 4(d).** Solution to BVP for Variable Forcing Function $f(y)$ ; *a=-1.*

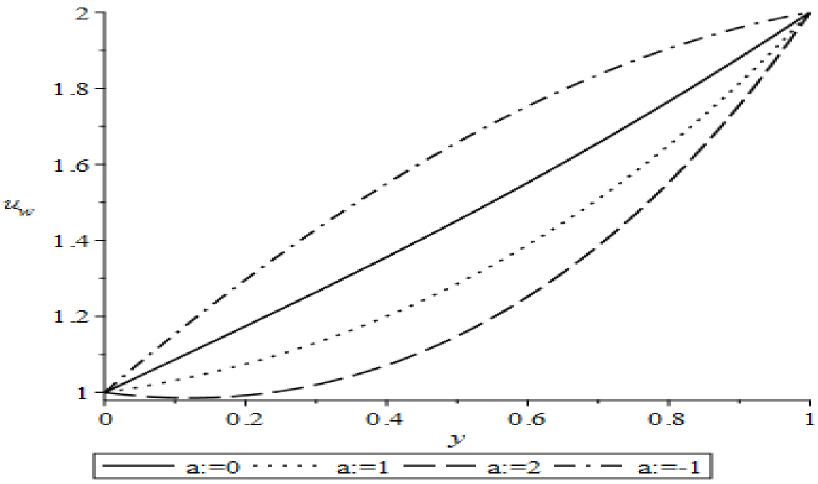


**Fig. 4(e).** Solution to BVP for $f(y) = y$ and Different Values of *a*.

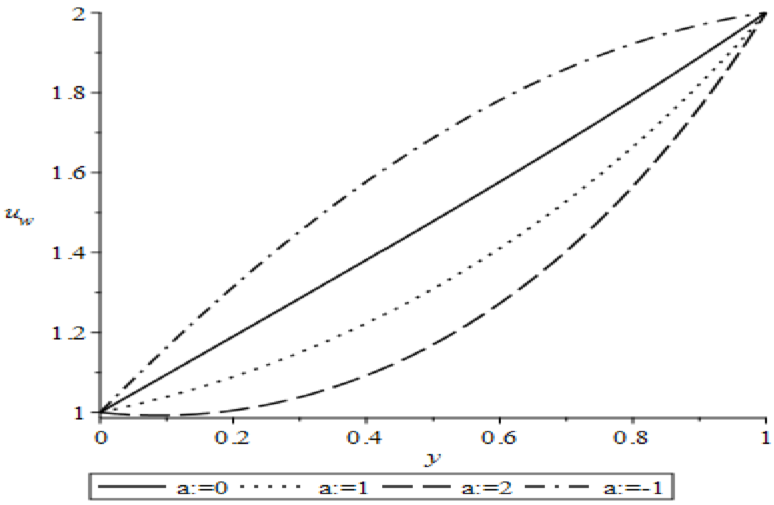


**Fig. 4(f).** Solution to BVP for $f(y) = y^2$ and Different Values of *a*.

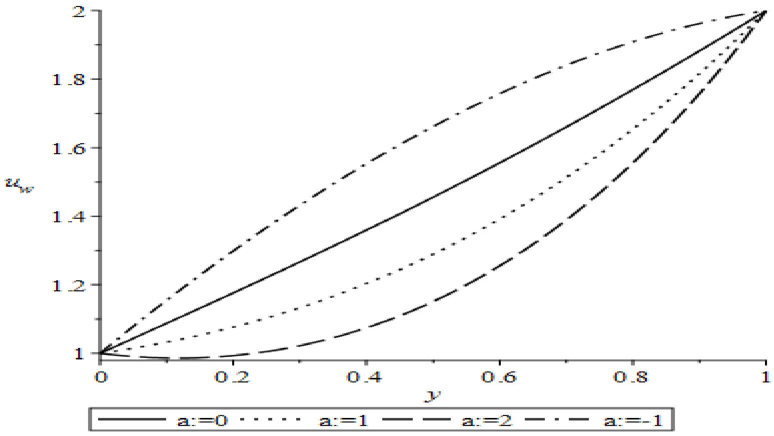


**Fig. 4(g).** Solution to BVP for $f(y) = \sin y$ and Different Values of *a*.

## VI. CONCLUSION

In this work we considered initial value and boundary value problems associated with Weber's inhomogeneous ordinary differential equation with real parameter *a*. Both cases of constant and variable forcing functions have been treated using five selected functions. Solutions have been expressed in terms of the parametric Nield-Kuznetsov functions of the first- and second-kind, and evaluated using computational procedures based on power series for the Weber and the Nield-Kuznetsov functions. Computations of the arbitrary constants and the solutions are tabulated or graphed in this work, and support the following conclusions.

(a) Values of the arbitrary constants in the case of initial value problem are independent of the forcing function in Weber's equation. Rather, they depend on the real parameter *a*.

(b) Values of the arbitrary constants in the case of boundary value problem depend on both the forcing function in Weber's equation and on the real parameter *a*.

(c) For the initial value problem with constant or variable forcing functions, the solution curve corresponding to $a = 0$ is an almost linear, increasing curve for all variable forcing functions tested. Above this curve lie the exponentially increasing solution curves corresponding to $a > 0$, and below which lies the parabolic curve that corresponds to $a < 0$.

(d) For the boundary value problem with constant or variable forcing functions, the solution curve corresponding to $a = 0$ is an almost linear, increasing curve. Above this curve lies the parabolic solution curve corresponding to $a < 0$, and below which lie the parabolic curve that corresponds to $a > 0$.

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### Biographies and Photographs

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