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Exact Shape Functions and stiffness Matrices of One-Dimensional Elements for Solution of Plate Problems

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ABSTRACT
The main propose of this article is to discuss an approximate but computationally manageable gridwork solution of plate problems as an application of finite element method. This method implies each discrete element can be defined as an extension of the discrete parameter approach based on the exact stiffness, obtained by exact shape functions for the beam element to extend for solution of complicate two-dimensional plate problems

Keywords *–Plates, griwork solution, stiffness, shape functions, Beam*
Keywords *- About five key words in alphabetical order, separated by commas.*

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I. INTRODUCTION

In many engineering structures assessment of stress conditions created by vertical or horizontal forces to the structural elements is a frequent problem of design. For particular plate problems, closed form solutions have been obtained. However, even for conventional plate analysis these solutions can usually be applied to the problems with simple geometry, load and boundary conditions. Some numerical and approximate methods, such as finite element, finite difference, boundary element and framework methods have been developed to overcome such problems. Much research has been conducted to deal with bending, buckling and vibration problems of beam and plates on elastic foundation. The aim of most is to solve plate problems such as structural foundation analysis of buildings, pavements of highways, water tanks, airport runways and buried pipelines, etc. Nowadays A broad range of the engineering problems has been solved by computer-based methods such as finite element and boundary element methods[1-4]. Closed form solutions of such complex plate problems have been published for a limited number of cases. On the other hand in the case of the beam analysis, the formulations based on interpolation (shape) functions have been used in solution by finite element method. In 1980's authors such as [5-7] have derived exact stiffness matrices for one-dimensional beam elements. The obtained matrices are extended to solution to an analytical solution for the shape functions of a beam segment supported on a generalized two-parameter elastic foundation[8]. In that study it is pointed out that the exact shape functions can be utilized to derive exact analytic expressions for the coefficients of the element stiffness matrix. The obtained matrices are very useful tool to obtain the way for deriving work equivalent nodal forces for arbitrary transverse loads and coefficients of the consistent mass and geometrical stiffness matrices.

This study let plates to be represented by a discrete number of intersecting beams. Thus, mechanical properties of one-dimensional beam elements are used for solution of complex plate problems for various loading and boundary conditions. As Wilson [9] has indicated the structural behaviour of a beam resembles that of a strip in a plate. On the other hand Hrennikof [10] stated that the system cannot truly be equal to the continuous structure. However it is shown that the ease in arriving at results of engineering accuracy outweighs the small errors that these results represent. Its errors are attributable to the torsional constants of the grid members and the compromised effects of discretizing a continuous problem [11]. Then the framework method that replaces a continuous surface by an idealized discrete system can represent a two-dimensional plate conceptualized[12,13]. By this representation, plate problems including non-uniform thickness and foundation properties, arbitrary boundary and loading conditions and discontinuous surfaces, can be solved in a general form. The theory of this study lets to replace the governing equation for plate bending problems expressed in Equation (1) which is quite a daunting equation to solve for general loads and boundary conditions by that of one-dimensional beam element in Equation (2). For the governing differential equations of a plate and a line element for transverse displacement w(x,y)and w(x) are given as;

$$D\left(\frac{\partial^4 w(x,y)}{\partial x^4}+2\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2}+\frac{\partial^4 w(x,y)}{\partial y^4}\right)=q(x,y) \qquad (1)$$

$$EI\frac{d^4 w(x)}{d^4 x}=q(x) \qquad (2)$$

where EI and D define the flexural rigidities of the one-dimensional beam element and that of two-dimensional plate element respectively. From the differential equations networks of beam elements that represent the plates has an obvious advantage to solve complicate plate problems.

II. DERIVATION OF EXACT SHAPE FUNCTIONS

For all types of loading and boundary conditions it is too difficult to obtain rigorous solution for two-dimensional plate problems. For most cases the series solutions are valid for limited cases such as simple loading and boundary conditions exist. As an alternative for different types of loading and boundary conditions it is possible to extend the solution for beam elements by the stiffness method for complex plate elements represented by a discrete number of intersecting beam elements. For solution of one-dimensional beam elements firstly derivation of shape functions can be obtained by homogeneous form of Equation (2) (let q(x)=0) and after necessary simplifications;

$$\frac{d^4 w(x)}{dx^4}=0 \qquad (3)$$

And by operator method using $\frac{d^n}{dx^n}=D^n$ then the characteristic Equation (3) can be written and the roots of the characteristic equation with imaginary number (i) are;

$$D^4 w(x)=0 \qquad (4)$$

Then the closed form solution of Equation (4) with inserting the angular displacements as Ø(x)= a_1+a_2x due to torsional effects with the procedures of the previous studies [14,15] by finite element based matrix methods the shape functions can be obtained. Than for flexure of uniform beam it is possible to find out the non-dimensional forms of the shape functions as Hermitian polynomials.

$$\psi_1=1-\xi$$
$$\frac{\psi_2}{L}=\xi-2\xi^2+\xi^3$$
$$\psi_3=3\xi^2-2\xi^3-1 \qquad \text{where} \quad \xi_1=\frac{1}{x} \qquad (5)$$
$$\psi_4=\xi$$
$$\frac{\psi_5}{L}=\xi^3+\xi^2$$
$$\psi_6=2\xi^3-3\xi^2$$

The exact shape functions obtained by Equation (5) are the main tool to determine the stiffness matrices of one-dimensional beam elements.

III. DERIVATION OF STIFFNESS MATRICES AND GRIDWORK

The element stiffness matrix relates the nodal forces to the nodal displacements for the prismatic beam element can be obtained from the minimization of strain energy functional U can be obtained by be multiplying Equation (2) with a test or weighting function, ν(x) which is a continuous function over the domain of the problem. The test function ν(x) viewed as a variation in w must be consistent with the boundary conditions. The variation in w as a virtual change vanishes at points where w is specified, and it is an arbitrary elsewhere. First step is to integrate the product over the domain;

$$\int_0^L \nu(x)\left[EI\frac{d^4w(x)}{dx^4} - q(x)\right]dx = 0 \quad \text{or} \quad \int_0^L \nu(x)e(x)dx = 0 \tag{6}$$

The purpose of the ν(x) is to minimize the function e(x), the residual of the differential equation, in weighted integral sense. Equation (6) is the weighted residual statement equivalent to the original differential equation. since v(x) is the variation in w(x), it has to satisfy homogeneous form of the essential boundary condition. Then, Equation (6) takes the form of only twice differentiable in contrast to Equation (2), which is in fourth order differential equation, as follows;

$$\int_0^L \nu(x)\left[EI\frac{d^4w(x)}{dx^4} - q(x)\right]dx = EI\int_0^L \frac{d^2\nu(x)}{dx^2}\frac{d^2w(x)}{dx^2}dx - \int_0^L \nu(x)q(x)dx = 0 \tag{7}$$

Equation (7) is called the weak, generalized or variational equation associated with Equation (2). The variational solution is not differentiable enough to satisfy the original differential equation. However it is differentiable enough to satisfy the variational equation equivalent to Equation (2). In order to obtain the stiffness matrix, the displacement fields can be defined as follows;

$$w(x) = \sum_{j=1}^{6}\psi_j w_j \quad and \quad \nu(x) = \psi_i \tag{8}$$

$$\begin{gathered} EI\int_0^L \frac{d^2\psi_i}{dx^2}\frac{d^2\psi_j}{dx^2}w_j dx - \int_0^L \psi_i q(x)dx = 0 \\ or \\ \left\{EI\int_0^L \frac{d^2\psi_i}{dx^2}\frac{d^2\psi_j}{dx^2}dx\right\}w_j = \int_0^L \psi_i q(x)dx \end{gathered} \tag{9}$$

and the end forces can be obtained as;

$$\{\underline{K_e}\}\{\underline{w_j}\} = \{\underline{F_e}\} \tag{10}$$

The shape functions, ψ1, ψ2, ψ3, ψ4, ψ5 and ψ6, are already obtained as determined in Equation (5). The nodal displacements are $\{\underline{w_j}\}^T = \{\phi_1, \theta_1, w_1, \phi_2, \theta_2, w_2\}$ referring to the sign convention and performing the necessary symbolic calculations, the terms of the stiffness matrix for the conventional beam stiffness terms obtained by Hermitian functions. Similar to determining the system stiffness matrix for bending, the system geometric stiffness and consistent mass matrices also can be obtained for buckling and vibration parameters. These individual element matrices can be used to form the system exact load and stiffness matrices for plates. In order to extend the stiffness terms of one-dimensional beam elements gridwork method is an applicable method for solution of plate problems. In this method, at edge nodes two or three, at interior nodes four of the typical discrete individual beam elements are intersected. However matrix displacement method based on stiffness-matrix approach is a useful tool to solve gridworks with arbitrary load and boundary conditions.

IV. CONCLUSION

The shape functions and the corresponded stiffness terms for one-dimensional beam elements can be used to form gridwork solution as a combination of finite element method, lattice analogy and matrix displacement analysis for more complicated plate problems. This solution lets the plate be modeled as an assemblage of individual beam elements interconnected at their neighboring joints to form continuous surface. The solution obtained for bending problems of the conventional beam elements can also extendable to consistent mass and geometric stiffness matrices for of beam elements as a valuable tools to solve plate vibration and buckling problems.

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