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Lie symmetries and classification of plane symmetric static space-times

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---***ABSTRACT***---
In this study, we provide the Lie symmetries and a classification of plane symmetric static space-times. Based on the invariance of the Lagrange equations of plane symmetries static space-time systems under the transformation Lie group, we provide the Lie symmetry determination equation, Lie symmetry theorem, and the conserved quantity of the systems; by utilizing theLie symmetry methodto solute the system, we give complete classification of the plane symmetric static space-times systems.The research results indicate that using the Lie symmetry method to study plane symmetric static spacetime can identify a series of conserved quantities that exist in the system; Discovered 5 basic Lie symmetries in the system; When the metric parameters of the system are appropriately selected, the plane symmetric static spacetime can have 6, 7, 8, 9, 11, and 17 Lie symmetric symmetries and conserved quantities.

Keywords: *static space-times; plane symmetric; Lie symmetry; conserved quantity; classification*

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I. Introduction

It is generally known that space-time symmetries play a significant role in the motion of particles, specifically in gravity theories. The classification of space-time symmetry has become a hot topic in general relativity [1-7].These classifications not only classify space-times according to such space-time symmetries, but also provide new solutions to the Einstein field equations, which are given in standard gravitational units $c = G = 1$ as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu}$$

The classification of Einstein field equations (EFE) constitutes a significant part of generalrelativity research. It is practically impossible to find general solutions for the EFE in a closed analytic form. These are non-linear partial differential equations, and it is not easy to obtain the exact solutions of these equations.

In 1974, Lie introduced infinitesimal transformations into differential equations and proposed a symmetric solution for solving differential equations [8]. The symmetry method is a fundamental method for solving differential equations [9-15]. By using the symmetry-solving method, we can solve the dynamical equations, reduce the order of the differential equations, and linearize nonlinear the dynamical equations. These methods are used to reduce the number of variables in partial differential equations. There are two basic symmetry methods under the transformation of the Lie group. One is based on the invariance of the Hamiltonian action of the dynamic system under the transformation Lie group, which is called the Noether symmetry method [16]; Other is based on the invariance of dynamical equations of system under the transformation Lie group, which is called the Lie symmetry method [8,16].

Symmetries help to find solutions of the 4 dimensional space-time. Various approaches have been used to classify space-times and to find solutions to Einstein field equations [17-23]. However, in previous studies, the classification and exact solutions of four-dimensional space-time were mostly based on the invariance of the Lagrange function describing the system under the transformation Lie group;that is, the Noether symmetry of a four-dimensional space-time system to give the conservation of the system's existence and classify the system.

In the theoretical study of symmetry, some important results have also been achieved in the classification of Lie symmetry in physics and mechanics [24-26].Tiwar et al studied Lie point symmetries classification of the mixed Liénard-type equation [24]. Baikov et al obtained Lie symmetry classification analysis for nonlinear coupled diffusion[25].Prince derived classification of dynamical symmetries in classical mechanics [26]. In recent years, the Lie symmetry method has been successfully applied to solve problems in conservative and non conservative, holonomic and non holonomic constrained mechanical systems, as well as in phase space constrained mechanical systems [27-41]. Scholars have applied the Lie symmetry method to solve electromechanical coupled dynamic systems and flexible robot systems [42-49].

In this study, theplane symmetric static space-times were classified using Lie symmetries. We introduce the concepts of generalized coordinates and generalized momentum in four-dimensional space-time, and provide the corresponding Lagrange equations. Based on the invariance of the Lagrange equations in the transformation Lie group of the system, we derive the Lie symmetry determination equation for the system, and further solve the transformation Lie group corresponding to the plane symmetric static space-time. We also propose the Lie symmetry theorem and conserved quantity (first integral) for plane symmetry static space-time. Lie symmetry theorem proved that for every symmetry there is a conservation law (conserved quantity). Classification of the plane symmetry space-times by Lie symmetries provides those symmetries of space-times corresponding to which there are conserved quantities. Isometries and homotheties also correspond to conserved quantities and they form subset(s) of the set of Lie symmetries. For the classification. examples of the Lie symmetries are symmetry under spatial translations implies conservation of linear momentums, the symmetry under time translation implies conservation of energy, and the symmetry under rotation implies conservation of angular momentum.

II. Motion equations of plane symmetrystatic space-time system

Infour-dimensional space,generally the line element of *n*th dimensional space-time takes the form

$$ds^2 = g_{ij} dx^i dx^j \quad i, j = 1, 2, \cdots, n \quad (1)$$

The Lagrangian *L* for the metric (1) is given by [23–26]

$$L = g_{ij} \dot{x}^i \dot{x}^j . \quad (2)$$

The spherically symmetric space-time is an important exact solution of the Einstein fieldequation. It is the conformally at solution of the Einstein-Maxwell field equations for anonnull electromagnetic field.We take the Lagrangian of general plane symmetric static space-time [7]

$$L = \frac{1}{2}\left[e^{\nu(x)} \dot{t}^2 - \dot{x}^2 - e^{\mu(x)} \left(\dot{y}^2 + \dot{z}^2 \right) \right] \quad (3)$$

where *s* is the independent variable, x^i are the dependent variables on *t*, *x*, y,*z*and $\dot{x}^i$ are theirderivatives with respect to curve *s*; and here themetric coefficients $\nu(x)$ and $\mu(x)$ are functions of coordinate *x*.

The static space-time of the plane symmetriescan be considered as a holonomicdynamical system. We can provethat one satisfies Lagrange equations in the following form:

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k \quad q_k = q_k\left(s, t, x, y, z\right). \quad (4)$$

The Lagrange equations (4) are different from the Lagrange equations in analytical mechanics, as it is the Lagrange equations in four-dimensional space-time.

Substituting Eq. (3) into Eqs. (4), we obtain the Lagrange equations for the plane symmetries static spacetime as follows

$$\ddot{t} = -\frac{d\nu}{dx}\dot{x}\dot{t} = \alpha_0 , \quad (5)$$

$$\ddot{x} = \frac{1}{2} e^{\nu(x)} \frac{d\nu}{dx} \dot{t}^2 - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx}\left(\dot{y}^2 + \dot{z}^2 \right) = \alpha_1 , \quad (6)$$

$$\ddot{y} = -\frac{d\mu}{dx}\dot{x}\dot{y} = \alpha_2 , \quad (7)$$

$$\ddot{z} = -\frac{d\mu}{dx}\dot{x}\dot{z} = \alpha_3 . \quad (8)$$

III. Lie symmetries and conserved quantities of the Plane symmetries static space-time

In this section, we perform infinitesimal transformations on the independent and dependent variables of the plane symmetries static space-time based on the invariance of the Lagrange equation of the plane symmetries static space-time under infinitesimal transformations.We provide the Lie symmetry determination equation, Lie symmetry theorem, and this system possesses the conserved quantity (first integral).

3.1 Lie symmetries and its determining equations of the plane symmetries static space-time

Introducing infinitesimal transformations on independent variable *s* and generalized coordinates,

$$s^* = s + \varepsilon\xi\left(s, q_0, q_1, \cdots, q_k\right), \quad q_k^* = q_k + \varepsilon\eta^k\left(s, q_0, q_1, \cdots, q_k\right) \quad k = 0, 1, \cdots, \ n \quad (9)$$

where ε is an infinitesimal parameter, ξ and η^k are infinitesimal generators, and n=4N-l . Introducing the following infinitesimal generator vector,

$$X^{[0]} = \xi \frac{\partial}{\partial s} + \sum_{k=0}^{n} \eta^k \frac{\partial}{\partial q_k}, \quad (10)$$

and, one of its extensions,

$$X^{[1]} = X^{[0]} + \sum_{k=0}^{n} \left(\dot{\eta}^k - \dot{q}_k \dot{\xi}\right) \frac{\partial}{\partial \dot{q}_k}, \quad (11)$$

and its secondary expansion,

$$X^{[2]} = X^{[1]} + \sum_{k=0}^{n} \left(\ddot{\eta}^k - 2\alpha_k \dot{\xi} - \dot{q}_k \ddot{\xi}\right) \frac{\partial}{\partial \dot{q}_k}. \quad (12)$$

In this work,we take K=t,x,y,z.

The invariance of differential equation (5)-(8) under infinitesimal transformation (9) is reduced to the following equations:

$$\ddot{\eta}^0 + 2\frac{d\nu}{dx}\dot{x}\dot{t}\dot{\xi} - \dot{t}\ddot{\xi} = -X^1\left(\frac{d\nu}{dx}\dot{x}\dot{t}\right), (13)$$

$$\ddot{\eta}^1 - \left(e^{\nu(x)}\frac{d\nu}{dx}\dot{t}^2 - e^{\mu(x)}\frac{d\mu}{dx}\left(\dot{y}^2 + \dot{z}^2\right)\right)\dot{\xi} - \dot{x}\ddot{\xi} = -X^1\left(\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\left(\dot{y}^2 + \dot{z}^2\right) - \frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\dot{t}^2\right), \quad (14)$$

$$\ddot{\eta}^2 + 2\frac{d\mu}{dx}\dot{x}\dot{y}\dot{\xi} - \dot{y}\ddot{\xi} = -X^1\left(\frac{d\mu}{dx}\dot{x}\dot{y}\right), \quad (15)$$

$$\ddot{\eta}^3 + 2\frac{d\mu}{dx}\dot{x}\dot{z}\dot{\xi} - \dot{z}\ddot{\xi} = -X^1\left(\frac{d\mu}{dx}\dot{x}\dot{z}\right). \quad (16)$$

Equations (13)-(16) are called the Lie symmetry determination equations for the plane symmetry static space-time inrectangular coordinate system.

By expanding Eqs.(13)-(16), we obtain the four partial differential equations oftheplane symmetries static space-timeas follows:

$$
\begin{aligned}
&\frac{\partial^2\eta^0}{\partial s^2}+\left(2\frac{\partial^2\eta^0}{\partial t\partial s}-\frac{\partial^2\xi}{\partial s^2}+\frac{\partial\eta^1}{\partial s}\frac{d\nu}{dx}\right)\dot t+\left(2\frac{\partial^2\eta^0}{\partial x\partial s}+\frac{\partial\eta^0}{\partial s}\frac{d\nu}{dx}\right)\dot x\\
&+2\frac{\partial^2\eta^0}{\partial y\partial s}\dot y+2\frac{\partial^2\eta^0}{\partial z\partial s}\dot z\\
&+\left(2\frac{\partial^2\eta^0}{\partial t\partial y}-2\frac{\partial^2\xi}{\partial y\partial s}+\frac{\partial\eta^1}{\partial y}\frac{d\nu}{dx}\right)\dot t\dot y+\left(2\frac{\partial^2\eta^0}{\partial t\partial z}-2\frac{\partial^2\xi}{\partial z\partial s}+\frac{\partial\eta^1}{\partial z}\frac{d\nu}{dx}\right)\dot t\dot z\\
&+\left(2\frac{\partial^2\eta^0}{\partial x\partial y}-\frac{d\mu}{dx}\frac{\partial\eta^0}{\partial y}+\frac{\partial\eta^0}{\partial y}\frac{d\nu}{dx}\right)\dot x\dot y+\left(2\frac{\partial^2\eta^0}{\partial x\partial z}-\frac{d\mu}{dx}\frac{\partial\eta^0}{\partial z}+\frac{\partial\eta^0}{\partial z}\frac{d\nu}{dx}\right)\dot x\dot z\\
&+2\frac{\partial^2\eta^0}{\partial z\partial y}\dot y\dot z\\
&+\left(\begin{aligned}&2\frac{d\nu}{dx}\frac{\partial\xi}{\partial s}-\frac{d\nu}{dx}\frac{\partial\eta^0}{\partial t}-2\frac{\partial^2\xi}{\partial x\partial s}+2\frac{\partial^2\eta^0}{\partial t\partial x}+\eta^1\frac{d^2\nu}{dx^2}+\frac{\partial\eta^0}{\partial t}\frac{d\nu}{dx}+\frac{\partial\eta^1}{\partial x}\frac{d\nu}{dx}\\&-2\frac{d\nu}{dx}\frac{\partial\xi}{\partial s}\end{aligned}\right)\dot x\dot t\\
&+\left(\frac{1}{2}e^{\nu(x)}\frac{\partial\eta^0}{\partial x}\frac{d\nu}{dx}+\frac{\partial^2\eta^0}{\partial t\partial t}-2\frac{\partial^2\xi}{\partial t\partial s}+\frac{\partial\eta^1}{\partial t}\frac{d\nu}{dx}\right)\dot t^2+\left(\frac{\partial\eta^0}{\partial x}\frac{d\nu}{dx}+\frac{\partial^2\eta^0}{\partial x\partial x}\right)\dot x^2\\
&+\left(\frac{\partial^2\eta^0}{\partial y\partial y}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^0}{\partial x}\right)\dot y^2+\left(\frac{\partial^2\eta^0}{\partial z\partial z}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^0}{\partial x}\right)\dot z^2\\
&+\left(-\frac{\partial^2\xi}{\partial t\partial t}\dot t^3-\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}\right)\dot t^3+\left(2\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial x\partial x}-2\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}\right)\dot t\dot x^2\\
&+\left(\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial y\partial y}\right)\dot t\dot y^2+\left(\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial z\partial z}\right)\dot t\dot z^2\\
&+\left(\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}-2\frac{\partial^2\xi}{\partial t\partial x}-2\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}+2\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}\right)\dot x\dot t^2-2\frac{\partial^2\xi}{\partial t\partial y}\dot y\dot t^2-2\frac{\partial^2\xi}{\partial t\partial z}\dot z\dot t^2\\
&+\left(\frac{d\mu}{dx}\frac{\partial\xi}{\partial y}-2\frac{\partial^2\xi}{\partial x\partial y}+2\frac{d\nu}{dx}\frac{\partial\xi}{\partial y}-2\frac{d\nu}{dx}\frac{\partial\xi}{\partial y}\right)\dot t\dot y\dot x\\
&+\left(2\frac{d\nu}{dx}\frac{\partial\xi}{\partial z}+\frac{d\mu}{dx}\frac{\partial\xi}{\partial z}-2\frac{\partial^2\xi}{\partial x\partial z}-2\frac{d\nu}{dx}\frac{\partial\xi}{\partial z}\right)\dot t\dot z\dot x-2\frac{\partial^2\xi}{\partial z\partial y}\dot t\dot y\dot z\\
&=0 \qquad (17)
\end{aligned}
$$

$$\frac{\partial^2\eta^0}{\partial s^2}=0,2\frac{\partial^2\eta^0}{\partial t\partial s}-\frac{\partial^2\xi}{\partial s^2}+\frac{\partial\eta^1}{\partial s}\frac{d\nu}{dx}=0,+2\frac{\partial^2\eta^0}{\partial x\partial s}+\frac{\partial\eta^0}{\partial s}\frac{d\nu}{dx}=0,$$
$$+2\frac{\partial^2\eta^0}{\partial y\partial s}=0,+2\frac{\partial^2\eta^0}{\partial z\partial s}=0,+2\frac{\partial^2\eta^0}{\partial t\partial y}-2\frac{\partial^2\xi}{\partial y\partial s}+\frac{\partial\eta^1}{\partial y}\frac{d\nu}{dx}=0,$$
$$+2\frac{\partial^2\eta^0}{\partial t\partial z}-2\frac{\partial^2\xi}{\partial z\partial s}+\frac{\partial\eta^1}{\partial z}\frac{d\nu}{dx}=0,+2\frac{\partial^2\eta^0}{\partial x\partial y}-\frac{d\mu}{dx}\frac{\partial\eta^0}{\partial y}+\frac{\partial\eta^0}{\partial y}\frac{d\nu}{dx}=0,$$
$$+2\frac{\partial^2\eta^0}{\partial x\partial z}-\frac{d\mu}{dx}\frac{\partial\eta^0}{\partial z}+\frac{\partial\eta^0}{\partial z}\frac{d\nu}{dx}=0,+2\frac{\partial^2\eta^0}{\partial z\partial y}=0$$
$$+2\frac{d\nu}{dx}\frac{\partial\xi}{\partial s}-\frac{d\nu}{dx}\frac{\partial\eta^0}{\partial t}-2\frac{\partial^2\xi}{\partial x\partial s}+2\frac{\partial^2\eta^0}{\partial t\partial x}+\eta^1\frac{d^2\nu}{dx^2}+\frac{\partial\eta^0}{\partial t}\frac{d\nu}{dx}$$
$$+\frac{\partial\eta^1}{\partial x}\frac{d\nu}{dx}-2\frac{d\nu}{dx}\frac{\partial\xi}{\partial s}=0,\frac{\partial\eta^0}{\partial x}\frac{d\nu}{dx}+\frac{\partial^2\eta^0}{\partial x\partial x}=0,$$
$$+\frac{1}{2}e^{\nu(x)}\frac{\partial\eta^0}{\partial x}\frac{d\nu}{dx}+\frac{\partial^2\eta^0}{\partial t\partial t}-2\frac{\partial^2\xi}{\partial t\partial s}+\frac{\partial\eta^1}{\partial t}\frac{d\nu}{dx}=0,$$
$$\frac{\partial^2\eta^0}{\partial y\partial y}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^0}{\partial x}=0,\frac{\partial^2\eta^0}{\partial z\partial z}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^0}{\partial x}=0,$$
$$-\frac{\partial^2\xi}{\partial t\partial t}\dot{t}^3-\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}=0,2\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial x\partial x}-2\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}=0,$$
$$\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial y\partial y}=0,\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial z\partial z}=0,$$
$$\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}-2\frac{\partial^2\xi}{\partial t\partial x}-2\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}+2\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}=0,-2\frac{\partial^2\xi}{\partial t\partial y}=0, \tag{17}$$
$$-2\frac{\partial^2\xi}{\partial t\partial z}=0,\frac{d\mu}{dx}\frac{\partial\xi}{\partial y}-2\frac{\partial^2\xi}{\partial x\partial y}+2\frac{d\nu}{dx}\frac{\partial\xi}{\partial y}-2\frac{d\nu}{dx}\frac{\partial\xi}{\partial y}=0,$$
$$2\frac{d\nu}{dx}\frac{\partial\xi}{\partial z}+\frac{d\mu}{dx}\frac{\partial\xi}{\partial z}-2\frac{\partial^2\xi}{\partial x\partial z}-2\frac{d\nu}{dx}\frac{\partial\xi}{\partial z}=0,-2\frac{\partial^2\xi}{\partial z\partial y}=0.$$

$$\frac{\partial^2\eta^1}{\partial s^2}+2\frac{\partial^2\eta^1}{\partial t\partial s}\dot{t}+\left(2\frac{\partial^2\eta^1}{\partial x\partial s}-\frac{\partial^2\xi}{\partial s^2}\right)\dot{x}+2\frac{\partial^2\eta^1}{\partial y\partial s}\dot{y}+2\frac{\partial^2\eta^1}{\partial z\partial s}\dot{z}$$

$$+\left(\frac{\partial^2\eta^1}{\partial t\partial t}+\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^1}{\partial x}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial s}\right)\dot{t}^2+\left(\frac{\partial^2\eta^1}{\partial x\partial x}-2\frac{\partial^2\xi}{\partial x\partial s}\right)\dot{x}^2$$

$$+\left(\frac{\partial^2\eta^1}{\partial y\partial y}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^1}{\partial x}+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial s}\right)\dot{y}^2$$

$$+\left(\frac{\partial^2\eta^1}{\partial z\partial z}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^1}{\partial x}+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial s}\right)\dot{z}^2$$

$$+\left(2\frac{\partial^2\eta^1}{\partial t\partial x}-\frac{d\nu}{dx}\frac{\partial\eta^1}{\partial t}-2\frac{\partial^2\xi}{\partial t\partial s}\right)\dot{x}\dot{t}+2\frac{\partial^2\eta^1}{\partial t\partial y}\dot{y}\dot{t}+2\frac{\partial^2\eta^1}{\partial t\partial z}\dot{z}\dot{t}$$

$$-2\frac{\partial^2\xi}{\partial y\partial s}\dot{x}\dot{y}-2\frac{\partial^2\xi}{\partial z\partial s}\dot{x}\dot{z}+\left(2\frac{\partial^2\eta^1}{\partial x\partial y}-\frac{d\mu}{dx}\frac{\partial\eta^1}{\partial y}\right)\dot{x}\dot{y}+\left(2\frac{\partial^2\eta^1}{\partial x\partial z}-\frac{d\mu}{dx}\frac{\partial\eta^1}{\partial z}\right)\dot{x}\dot{z}$$

$$+2\frac{\partial^2\eta^1}{\partial z\partial y}\dot{y}\dot{z}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}\dot{t}^3-\frac{\partial^2\xi}{\partial x\partial x}\dot{x}^3+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial y}\dot{y}^3+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial z}\dot{z}^3$$

$$+\left(-\frac{\partial^2\xi}{\partial t\partial t}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}\right)\dot{t}^2\dot{x}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial y}\dot{t}^2\dot{y}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial z}\dot{t}^2\dot{z}$$

$$+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial t}\dot{t}\dot{y}^2+\left(e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial y\partial y}+\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}\right)\dot{x}\dot{y}^2+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial z}\dot{y}^2\dot{z}$$

$$+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial t}\dot{t}\dot{z}^2+\left(e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial z\partial z}+\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}\right)\dot{x}\dot{z}^2+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial y}\dot{y}\dot{z}^2$$

$$+\left(\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}-2\frac{\partial^2\xi}{\partial t\partial x}\right)\dot{x}^2\dot{t}-\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}\dot{x}\dot{t}^2$$

$$+\left(\frac{d\mu}{dx}\frac{\partial\xi}{\partial z}-2\frac{\partial^2\xi}{\partial x\partial z}\right)\dot{z}\dot{x}^2+\left(\frac{d\mu}{dx}\frac{\partial\xi}{\partial y}-2\frac{\partial^2\xi}{\partial x\partial y}\right)\dot{x}^2\dot{y}$$

$$-2\frac{\partial^2\xi}{\partial t\partial y}\dot{x}\dot{y}\dot{t}-2\frac{\partial^2\xi}{\partial t\partial z}\dot{x}\dot{z}\dot{t}-2\frac{\partial^2\xi}{\partial z\partial y}\dot{x}\dot{y}\dot{z}$$

$$\frac{\partial^2\eta^1}{\partial s^2}+\left(2\frac{\partial^2\eta^1}{\partial t\partial s}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^0}{\partial s}\right)\dot{t}+\left(2\frac{\partial^2\eta^1}{\partial x\partial s}-\frac{\partial^2\xi}{\partial s^2}\right)\dot{x}$$

$$+\left(2\frac{\partial^2\eta^1}{\partial y\partial s}+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial s}\right)\dot{y}+\left(2\frac{\partial^2\eta^1}{\partial z\partial s}+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial s}\right)\dot{z}$$

$$+\left(\begin{array}{l}\frac{\partial^2\eta^1}{\partial t\partial t}+\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^1}{\partial x}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial s}-\frac{1}{2}e^{\nu(x)}\left(\frac{d\nu}{dx}\right)^2\eta^1\\-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^0}{\partial t}+e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial s}\end{array}\right)\dot{t}^2$$

$$+\left(\frac{\partial^2\eta^1}{\partial x\partial x}-2\frac{\partial^2\xi}{\partial x\partial s}\right)\dot{x}^2$$

$$+\left(\begin{array}{l}\frac{\partial^2\eta^1}{\partial y\partial y}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^1}{\partial x}+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial s}+\frac{1}{2}e^{\mu(x)}\left(\frac{d\mu}{dx}\right)^2\eta^1\\+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial y}-e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial s}\end{array}\right)\dot{y}^2$$

$$+\left(\begin{array}{l}\frac{\partial^2\eta^1}{\partial z\partial z}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^1}{\partial x}+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial s}+\frac{1}{2}e^{\mu(x)}\left(\frac{d\mu}{dx}\right)^2\eta^1\\+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial z}-e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial s}\end{array}\right)\dot{z}^2 \qquad (18)$$

$$\frac{\partial^2\eta^1}{\partial s^2}=0, 2\frac{\partial^2\eta^1}{\partial t\partial s}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^0}{\partial s}=0, 2\frac{\partial^2\eta^1}{\partial x\partial s}-\frac{\partial^2\xi}{\partial s^2}=0,$$

$$2\frac{\partial^2\eta^1}{\partial y\partial s}+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial s}=0, 2\frac{\partial^2\eta^1}{\partial z\partial s}+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial s}=0,$$

$$\frac{\partial^2\eta^1}{\partial t\partial t}+\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^1}{\partial x}-\frac{1}{2}e^{\nu(x)}\left(\frac{d\nu}{dx}\right)^2\eta^1-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^0}{\partial t}=0,$$

$$\frac{\partial^2\eta^1}{\partial y\partial y}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^1}{\partial x}+\frac{1}{2}e^{\mu(x)}\left(\frac{d\mu}{dx}\right)^2\eta^1+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial y}=0,$$

$$\frac{\partial^2\eta^1}{\partial z\partial z}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^1}{\partial x},+\frac{1}{2}e^{\mu(x)}\left(\frac{d\mu}{dx}\right)^2\eta^1+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial z}=0,$$

$$2\frac{\partial^2\eta^1}{\partial t\partial x}-\frac{d\nu}{dx}\frac{\partial\eta^1}{\partial t}-2\frac{\partial^2\xi}{\partial t\partial s}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^0}{\partial x}=0,$$

$$e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial x}-2\frac{\partial^2\xi}{\partial y\partial s}+2\frac{\partial^2\eta^1}{\partial x\partial y}-\frac{d\mu}{dx}\frac{\partial\eta^1}{\partial y}=0,$$

$$e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial x}-2\frac{\partial^2\xi}{\partial z\partial s}+2\frac{\partial^2\eta^1}{\partial x\partial z}-\frac{d\mu}{dx}\frac{\partial\eta^1}{\partial z}=0,$$

$$e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial t}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^0}{\partial z}+2\frac{\partial^2\eta^1}{\partial t\partial z}=0,$$

$$e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial t}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^0}{\partial y}+2\frac{\partial^2\eta^1}{\partial t\partial y}=0,$$

$$e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial z}+e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial y}+2\frac{\partial^2\eta^1}{\partial z\partial y}=0,$$

$$e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}=0,-\frac{\partial^2\xi}{\partial x\partial x}=0,\frac{\partial^2\eta^1}{\partial x\partial x}-2\frac{\partial^2\xi}{\partial x\partial s}=0,$$

$$-\frac{\partial^2\xi}{\partial t\partial t}-e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}+\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}=0, \quad ,\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}-2\frac{\partial^2\xi}{\partial t\partial x}=0,$$

$$\frac{d\mu}{dx}\frac{\partial\xi}{\partial z}-2\frac{\partial^2\xi}{\partial x\partial z}=0, \quad e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial t}-e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial t}=0,\frac{d\mu}{dx}\frac{\partial\xi}{\partial y}-2\frac{\partial^2\xi}{\partial x\partial y}=0,$$

$$e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial y\partial y}+\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}-e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}=0,$$

$$e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial z}-e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial z}=0, \quad e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial y}-e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial y}=0, \qquad (18)$$

$$-\frac{\partial^2\xi}{\partial z\partial z}+\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}=0,-2\frac{\partial^2\xi}{\partial t\partial y}=0,2\frac{\partial^2\xi}{\partial t\partial z}=0,-2\frac{\partial^2\xi}{\partial z\partial y}=0.$$

$$\frac{\partial^2\eta^2}{\partial s\partial s}+2\frac{\partial^2\eta^2}{\partial s\partial t}\dot{t}+2\frac{\partial^2\eta^2}{\partial s\partial x}\dot{x}+\left(2\frac{\partial^2\eta^2}{\partial s\partial y}-\frac{\partial^2\xi}{\partial s^2}+\frac{\partial\eta^1}{\partial s}\frac{d\mu}{dx}+\frac{\partial\eta^2}{\partial s}\frac{d\mu}{dx}\right)\dot{y}+2\frac{\partial^2\eta^2}{\partial s\partial z}\dot{z}$$
$$+\left(\frac{\partial^2\eta^2}{\partial t\partial t}+\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^2}{\partial x}-\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}\right)\dot{t}^2+\frac{\partial^2\eta^2}{\partial x\partial x}\dot{x}^2$$
$$+\left(\frac{\partial^2\eta^2}{\partial x\partial y}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial x}-2\frac{\partial^2\xi}{\partial y\partial s}+\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}+\frac{\partial\eta^1}{\partial y}\frac{d\mu}{dx}+\frac{\partial\eta^2}{\partial y}\frac{d\mu}{dx}\right)\dot{y}^2$$
$$+\left(\frac{\partial^2\eta^2}{\partial z\partial z}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial x}+\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}\right)\dot{z}^2+\left(2\frac{\partial^2\eta^2}{\partial t\partial x}-\frac{d\nu}{dx}\frac{\partial\eta^2}{\partial t}+\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}\right)\dot{x}\dot{t}$$
$$+\left(2\frac{\partial^2\eta^2}{\partial t\partial y}-2\frac{\partial^2\xi}{\partial t\partial s}+\frac{\partial\eta^1}{\partial t}\frac{d\mu}{dx}+\frac{\partial\eta^2}{\partial t}\frac{d\mu}{dx}\right)\dot{t}\dot{y}+2\frac{\partial^2\eta^2}{\partial t\partial z}\dot{t}\dot{z}$$
$$+\left(\begin{array}{l}2\frac{\partial^2\eta^2}{\partial x\partial y}-\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial y}+2\frac{\partial\xi}{\partial s}\frac{d\mu}{dx}-2\frac{\partial^2\xi}{\partial x\partial s}+\frac{d\mu}{dx}\frac{\partial\xi}{\partial y}-2\frac{\partial\xi}{\partial s}\frac{d\mu}{dx}+\eta^1\frac{d^2\mu}{dx^2}\\+\frac{\partial\eta^1}{\partial x}\frac{d\mu}{dx}+\frac{\partial\eta^2}{\partial x}\frac{d\mu}{dx}\end{array}\right)\dot{x}\dot{y}$$
$$+\left(2\frac{\partial^2\eta^2}{\partial x\partial z}-\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial z}+\frac{d\mu}{dx}\frac{\partial\xi}{\partial z}\right)\dot{x}\dot{z}+\left(2\frac{\partial^2\eta^2}{\partial y\partial z}-2\frac{\partial^2\xi}{\partial z\partial s}+\frac{\partial\eta^1}{\partial z}\frac{d\mu}{dx}+\frac{\partial\eta^2}{\partial z}\frac{d\mu}{dx}\right)\dot{y}\dot{z}$$
$$+\left(2\frac{\partial\xi}{\partial t}\frac{d\mu}{dx}-2\frac{\partial^2\xi}{\partial t\partial x}-2\frac{\partial\xi}{\partial t}\frac{d\mu}{dx}\right)\dot{x}\dot{t}\dot{y}+\left(2\frac{\partial\xi}{\partial x}\frac{d\mu}{dx}-\frac{\partial^2\xi}{\partial x\partial x}-2\frac{\partial\xi}{\partial x}\frac{d\mu}{dx}\right)\dot{x}^2\dot{y}$$
$$+\left(2\frac{\partial\xi}{\partial y}\frac{d\mu}{dx}-2\frac{\partial^2\xi}{\partial x\partial y}-2\frac{\partial\xi}{\partial y}\frac{d\mu}{dx}\right)\dot{y}^2\dot{x} \qquad (19)$$
$$+\left(2\frac{\partial\xi}{\partial z}\frac{d\mu}{dx}-2\frac{\partial^2\xi}{\partial x\partial z}-2\frac{\partial\xi}{\partial z}\frac{d\mu}{dx}\right)\dot{z}\dot{x}\dot{y}-\frac{\partial^2\xi}{\partial t\partial t}\dot{t}^2\dot{y}-\frac{\partial^2\xi}{\partial z\partial z}\dot{z}^2\dot{y}$$
$$-2\frac{\partial^2\xi}{\partial t\partial y}\dot{y}^2\dot{t}-2\frac{\partial^2\xi}{\partial t\partial z}\dot{y}\dot{z}\dot{t}-\frac{\partial^2\xi}{\partial y\partial y}\dot{y}^3-2\frac{\partial^2\xi}{\partial z\partial y}\dot{y}^2\dot{z}=0$$

$$\frac{\partial^2\eta^2}{\partial s\partial s}=0,+2\frac{\partial^2\eta^2}{\partial s\partial t}=0,2\frac{\partial^2\eta^2}{\partial s\partial x}=0,2\frac{\partial^2\eta^2}{\partial s\partial y}-\frac{\partial^2\xi}{\partial s^2}+\frac{\partial\eta^1}{\partial s}\frac{d\mu}{dx}+\frac{\partial\eta^2}{\partial s}\frac{d\mu}{dx}=0,$$

$$\frac{\partial^2\eta^2}{\partial s\partial z}=0,\frac{\partial^2\eta^2}{\partial t\partial t}+\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^2}{\partial x}-\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}=0,+\frac{\partial^2\eta^2}{\partial x\partial x}=0,$$

$$\frac{\partial^2\eta^2}{\partial x\partial y}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial x}-2\frac{\partial^2\xi}{\partial y\partial s}+\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}+\frac{\partial\eta^1}{\partial y}\frac{d\mu}{dx}+\frac{\partial\eta^2}{\partial y}\frac{d\mu}{dx}=0,$$

$$\frac{\partial^2\eta^2}{\partial z\partial z}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial x}+\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}=0,2\frac{\partial^2\eta^2}{\partial t\partial x}-\frac{d\nu}{dx}\frac{\partial\eta^2}{\partial t}+\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}=0,$$

$$2\frac{\partial^2\eta^2}{\partial t\partial y}-2\frac{\partial^2\xi}{\partial t\partial s}+\frac{\partial\eta^1}{\partial t}\frac{d\mu}{dx}+\frac{\partial\eta^2}{\partial t}\frac{d\mu}{dx}=0,+2\frac{\partial^2\eta^2}{\partial t\partial z}=0,$$

$$2\frac{\partial^2\eta^2}{\partial x\partial y}-\frac{d\mu}{dx}\frac{\partial\eta^2}{\partial y}-2\frac{\partial^2\xi}{\partial x\partial s}+\frac{d\mu}{dx}\frac{\partial\xi}{\partial y}+\eta^1\frac{d^2\mu}{dx^2}+\frac{\partial\eta^1}{\partial x}\frac{d\mu}{dx}+\frac{\partial\eta^2}{\partial x}\frac{d\mu}{dx}=0,$$

$$2\frac{\partial^2\eta^2}{\partial x\partial z}+\frac{d\mu}{dx}\frac{\partial\xi}{\partial z}=0,2\frac{\partial^2\eta^2}{\partial y\partial z}-2\frac{\partial^2\xi}{\partial z\partial s}+\frac{\partial\eta^1}{\partial z}\frac{d\mu}{dx}=0,$$

$$-2\frac{\partial^2\xi}{\partial t\partial x}=0,\frac{\partial^2\xi}{\partial x\partial x}=0,2\frac{\partial^2\xi}{\partial x\partial y}=0,$$

$$2\frac{\partial\xi}{\partial z}\frac{d\mu}{dx}-2\frac{\partial^2\xi}{\partial x\partial z}-2\frac{\partial\xi}{\partial z}\frac{d\mu}{dx}=0,\frac{\partial^2\xi}{\partial t\partial t}=0,\frac{\partial^2\xi}{\partial z\partial z}=0, \tag{19}$$

$$\frac{\partial^2\xi}{\partial t\partial y}=0,\frac{\partial^2\xi}{\partial t\partial z}=0,\frac{\partial^2\xi}{\partial y\partial y}=0,\frac{\partial^2\xi}{\partial z\partial y}=0$$

$$\frac{\partial^2\eta^3}{\partial s^2}+2\frac{\partial^2\eta^3}{\partial t\partial s}\dot{t}+\left(2\frac{\partial^2\eta^3}{\partial x\partial s}+\frac{\partial\eta^3}{\partial s}\frac{d\mu}{dx}\right)\dot{x}+2\frac{\partial^2\eta^3}{\partial y\partial s}\dot{y}+\left(2\frac{\partial^2\eta^3}{\partial z\partial s}-\frac{\partial^2\xi}{\partial s^2}+\frac{\partial\eta^1}{\partial s}\frac{d\mu}{dx}\right)\dot{z}$$
$$+\left(\frac{\partial^2\eta^3}{\partial t\partial t}+\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^3}{\partial x}\right)\dot{t}^2+\left(\frac{\partial^2\eta^3}{\partial x\partial x}+\frac{\partial\eta^3}{\partial x}\frac{d\mu}{dx}\right)\dot{x}^2$$
$$+\left(\frac{\partial^2\eta^3}{\partial y\partial y}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial x}\right)\dot{y}^2$$
$$+\left(-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial x}-2\frac{\partial^2\xi}{\partial z\partial s}+\frac{\partial\eta^1}{\partial z}\frac{d\mu}{dx}+\frac{\partial^2\eta^3}{\partial z\partial z}\right)\dot{z}^2$$
$$+\left(2\frac{\partial^2\eta^3}{\partial t\partial x}-\frac{d\nu}{dx}\frac{\partial\eta^3}{\partial t}+\frac{\partial\eta^3}{\partial t}\frac{d\mu}{dx}\right)\dot{x}\dot{t}+2\frac{\partial^2\eta^3}{\partial t\partial y}\dot{y}\dot{t}+\left(2\frac{\partial^2\eta^3}{\partial t\partial z}-2\frac{\partial^2\xi}{\partial t\partial s}+\frac{\partial\eta^1}{\partial t}\frac{d\mu}{dx}\right)\dot{t}\dot{z}$$
$$+\left(2\frac{\partial^2\eta^3}{\partial x\partial y}-\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial y}+\frac{\partial\eta^3}{\partial y}\frac{d\mu}{dx}\right)\dot{x}\dot{y}$$
$$+\begin{pmatrix}2\frac{\partial^2\eta^3}{\partial x\partial z}-\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial z}+2\frac{\partial\xi}{\partial s}\frac{d\mu}{dx}+\eta^1\frac{d^2\mu}{dx^2}-2\frac{\partial\xi}{\partial s}\frac{d\mu}{dx}\\ -2\frac{\partial^2\xi}{\partial x\partial s}+\frac{\partial\eta^1}{\partial x}\frac{d\mu}{dx}+\frac{\partial\eta^3}{\partial z}\frac{d\mu}{dx}\end{pmatrix}\dot{x}\dot{z}$$
$$+\left(2\frac{\partial^2\eta^3}{\partial z\partial y}-2\frac{\partial^2\xi}{\partial y\partial s}+\frac{\partial\eta^1}{\partial y}\frac{d\mu}{dx}\right)\dot{y}\dot{z}$$
$$+\left(2\frac{\partial\xi}{\partial t}\frac{d\mu}{dx}-2\frac{\partial^2\xi}{\partial t\partial x}+\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}-2\frac{\partial\xi}{\partial t}\frac{d\mu}{dx}\right)\dot{x}\dot{t}\dot{z}-2\frac{\partial^2\xi}{\partial t\partial y}\dot{y}\dot{t}\dot{z}$$
$$+\left(2\frac{\partial\xi}{\partial x}\frac{d\mu}{dx}-\frac{\partial^2\xi}{\partial x\partial x}-2\frac{\partial\xi}{\partial x}\frac{d\mu}{dx}\right)\dot{x}^2\dot{z}$$
$$+\left(2\frac{\partial\xi}{\partial y}\frac{d\mu}{dx}-2\frac{\partial^2\xi}{\partial x\partial y}+\frac{d\mu}{dx}\frac{\partial\xi}{\partial y}-2\frac{\partial\xi}{\partial y}\frac{d\mu}{dx}+\frac{d\mu}{dx}\frac{\partial\xi}{\partial z}\right)\dot{y}\dot{x}\dot{z}$$
$$+\left(\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial y\partial y}\right)\dot{y}^2\dot{z}-2\frac{\partial^2\xi}{\partial z\partial y}\dot{y}\dot{z}^2-2\frac{\partial^2\xi}{\partial t\partial z}\dot{z}^2\dot{t}$$
$$+\left(-\frac{\partial^2\xi}{\partial t\partial t}-\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}\right)\dot{z}\dot{t}^2+\left(2\frac{\partial\xi}{\partial z}\frac{d\mu}{dx}-2\frac{\partial^2\xi}{\partial x\partial z}-2\frac{\partial\xi}{\partial z}\frac{d\mu}{dx}\right)\dot{z}^2\dot{x}$$
$$+\left(\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial z\partial z}\right)\dot{z}^3=0 \qquad (20)$$

$$\frac{\partial^2\eta^3}{\partial s^2}=0, 2\frac{\partial^2\eta^3}{\partial t\partial s}=0, 2\frac{\partial^2\eta^3}{\partial x\partial s}+\frac{\partial\eta^3}{\partial s}\frac{d\mu}{dx}=0, 2\frac{\partial^2\eta^3}{\partial y\partial s}=0,$$
$$2\frac{\partial^2\eta^3}{\partial z\partial s}-\frac{\partial^2\xi}{\partial s^2}+\frac{\partial\eta^1}{\partial s}\frac{d\mu}{dx}=0, \frac{\partial^2\eta^3}{\partial t\partial t}+\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\eta^3}{\partial x}=0,$$
$$\frac{\partial^2\eta^3}{\partial x\partial x}+\frac{\partial\eta^3}{\partial x}\frac{d\mu}{dx}=0, \frac{\partial^2\eta^3}{\partial y\partial y}-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial x}=0,$$
$$-\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\eta^3}{\partial x}-2\frac{\partial^2\xi}{\partial z\partial s}+\frac{\partial\eta^1}{\partial z}\frac{d\mu}{dx}+\frac{\partial^2\eta^3}{\partial z\partial z}=0,$$
$$2\frac{\partial^2\eta^3}{\partial t\partial x}-\frac{d\nu}{dx}\frac{\partial\eta^3}{\partial t}+\frac{\partial\eta^3}{\partial t}\frac{d\mu}{dx}=0, 2\frac{\partial^2\eta^3}{\partial t\partial y}=0,$$
$$2\frac{\partial^2\eta^3}{\partial t\partial z}-2\frac{\partial^2\xi}{\partial t\partial s}+\frac{\partial\eta^1}{\partial t}\frac{d\mu}{dx}=0, 2\frac{\partial^2\eta^3}{\partial x\partial y}=0,$$
$$2\frac{\partial^2\eta^3}{\partial x\partial z}+\eta^1\frac{d^2\mu}{dx^2}-2\frac{\partial^2\xi}{\partial x\partial s}+\frac{\partial\eta^1}{\partial x}\frac{d\mu}{dx}=0,$$
$$2\frac{\partial^2\eta^3}{\partial z\partial y}-2\frac{\partial^2\xi}{\partial y\partial s}+\frac{\partial\eta^1}{\partial y}\frac{d\mu}{dx}=0, -2\frac{\partial^2\xi}{\partial t\partial x}+\frac{d\nu}{dx}\frac{\partial\xi}{\partial t}=0,$$
$$-2\frac{\partial^2\xi}{\partial t\partial y}=0, 3\frac{\partial\xi}{\partial y}\frac{d\mu}{dx}-2\frac{\partial^2\xi}{\partial x\partial y}-2\frac{\partial\xi}{\partial y}\frac{d\mu}{dx}+\frac{d\mu}{dx}\frac{\partial\xi}{\partial z}=0,$$
$$\frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial y\partial y}=0, \frac{\partial^2\xi}{\partial z\partial y}=0, \frac{\partial^2\xi}{\partial t\partial z}=0, \frac{\partial^2\xi}{\partial x\partial x}=0, \tag{20}$$
$$-\frac{\partial^2\xi}{\partial t\partial t}-\frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\frac{\partial\xi}{\partial x}=0, 2\frac{\partial^2\xi}{\partial x\partial z}=0, \frac{1}{2}e^{\mu(x)}\frac{d\mu}{dx}\frac{\partial\xi}{\partial x}-\frac{\partial^2\xi}{\partial z\partial z}=0.$$

3.2 Five Lie symmetry generators and vector fields of plane symmetry static space-time

This system consists of nine unknowns ξ and η^i (i= 0, 1, 2, 3), ν, μ, λ, and G. Solutions of this system give the Lagrange equations along with the Lie symmetry corresponding to these Lagrange equations.One can easily write plane symmetries space-time, which are the exact solutions of the EFE. The Eqs.(17)-(20) yield the following solutions:

$$\xi=1, \eta^0=\eta^1=\eta^2=\eta^3=0, \quad Y_0=\frac{\partial}{\partial s}; \tag{21}$$
$$\eta^0=1, \xi=\eta^1=\eta^2=\eta^3=0, \quad X_0=\frac{\partial}{\partial t}; \tag{22}$$
$$\eta^2=1, \xi=\eta^0=\eta^1=\eta^3=0, \quad X_2=\frac{\partial}{\partial y}; \tag{23}$$
$$\eta^3=1, \xi=\eta^0=\eta^1=\eta^2=0, \quad X_3=\frac{\partial}{\partial z}; \tag{24}$$
$$\eta^3=y, \eta^2=-z, \xi=\eta^0=\eta^1=0, \quad X_3=y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}. \tag{25}$$

In other words, when the metric parameter ν, μ, λ take any value, the planesymmetry static space-timespossess at least five basic Lie symmetries.

3.3Lie symmetry theorem of plane symmetrystatic space-time

Theorem: for the generators $\xi, \eta^0, \eta^1, \eta^2, \eta^3$ of infinitesimal transformation forplane symmetry static space-time that satisfies determining equations (13)-(16), if there is a gauge function $G(s,t,x,y,z)$, satisfies the following equation:

$$L\dot{\xi}+X^{(1)}(L)+\sum_{k=0}^{3}Q_k\left(\dot{\eta}^k-\dot{q}_k\dot{\xi}\right)+\dot{G}=0, \qquad (26)$$

then the plane static spacetime possesses the conserved quantity:

$$I=L\xi+\sum_{k=0}^{3}\frac{\partial L}{\partial\dot{q}_k}\left(\eta^k-\dot{q}_k\xi\right)+G=\text{const}, \qquad (27)$$

where the Q_k are non-potential generalized forces in a plane symmetry static space-time. In this study, we take Q_k=0.

Proof:

$$\frac{\mathrm{d}I}{\mathrm{d}s}=\dot{L}\xi+L\dot{\xi}+\sum_{k=0}^{3}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial\dot{q}_k}\left(\eta^k-\dot{q}_k\xi\right)+\sum_{k=0}^{3}\frac{\partial L}{\partial\dot{q}_k}\left(\dot{\eta}^k-\ddot{q}_k\xi-\dot{q}_k\dot{\xi}\right)+\dot{G},$$

Using Eq.(25) in this equation and making further simplification, we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}s}=\sum_{k=1}^{4}\left(\xi^k-\dot{q}_k\xi_0\right)\left(\frac{\mathrm{d}L}{\mathrm{d}s}\frac{\partial L}{\partial\dot{q}_k}-\frac{\partial L}{\partial q_k}-Q_K\right)=0.$$

Equation (26) is called the structural equation of Lie symmetries of a plane symmetry static space-time.

3.4 Conserved quantities of the plane symmetry static space-time

The set Eqs.(20) –(24) of Lie symmetries for plane symmetry static space-time with a constant value of the gauge function. This is the minimal set of Lie symmetries for plane symmetric static space-times. If the metric parameters $\nu,\ \mu,\ \lambda$ of this system are obtain in the following form:

(a) $\nu(x)=\ln\left(\frac{x}{a}\right)^2,\quad \mu(x)=\left(\frac{x}{b}\right);$

(b) $\nu(x)=\left(\frac{x}{b}\right),\quad \mu(x)=\ln\left(\frac{x}{a}\right)^2;$

(c) $\nu(x)=\left(\frac{x}{a}\right)^2,\quad \mu(x)=\ln\cosh^2\left(\frac{x}{b}\right);$

(d) $\nu(x)=\left(\frac{x}{a}\right)^2,\quad \mu(x)=\ln\cos^2\left(\frac{x}{b}\right);$

(e) $\nu(x)=\ln\cosh^2\left(\frac{x}{b}\right),\quad \mu(x)=\left(\frac{x}{a}\right)^2;$

(f) $\nu(x)=\ln\cos^2\left(\frac{x}{b}\right),\quad \mu(x)=\left(\frac{x}{a}\right)^2;$

(g) $\nu(x)=\ln\cosh^2\left(\frac{x}{b}\right),\quad \mu(x)=\left(\frac{x}{a}\right);$

(h) $\nu(x)=\ln\cos^2\left(\frac{x}{b}\right),\quad \mu(x)=\left(\frac{x}{a}\right):$

(i) $\nu'(x)\neq 0,\nu(x)\neq\mu(x),\quad \mu(x)=\ln\cosh^2\left(\frac{x}{a}\right),\ln\cos^2\left(\frac{x}{a}\right);$

(j) $\mu(x)=\ln\cosh^2\left(\frac{x}{a}\right),\ln\cos^2\left(\frac{x}{a}\right),\quad \nu(x)\neq\mu(x),\mu'(x)\neq 0;$

(k) $\nu''(x)\neq 0,\quad \mu(x)=a\ln\frac{x}{a};$

(l) $\nu(x) = 2\ln\left(\frac{x}{a}\right), \quad \mu''(x) \neq 0, \mu(x) \neq a\ln\frac{x}{a}$

(m) $\nu(x) \neq \mu(x), \nu'(x) \neq 0, \quad \mu''(x) \neq 0, \mu(x) \neq a\ln\frac{x}{a};$

(n) $\nu''(x) \neq 0, \nu(x) \neq a\ln\frac{x}{a}, \quad \nu(x) \neq \mu(x), \mu'(x) \neq 0.$

By using Lie's theorem, we can provide conserved quantities for the existence of a plane symmetrystatic space-time and provide an accurate solution for the system.
By substituting generators (21)-(25) into (26) and (27) respectively, we obtain that the plane symmetry static space-time possesses conserved quantities in the following forms:

$$\begin{aligned} &X_0: \quad I_0 = -e^{\nu(x)}\dot{t} = \text{const}; \\ &X_1: \quad I_2 = e^{\mu(x)}\dot{y} = \text{const}; \\ &X_2: \quad I_3 = e^{\mu(x)}\dot{z} = \text{const}; \\ &\text{X}_3: \quad I_4 = e^{\mu(x)}\left(z\dot{y} - y\dot{z}\right) = \text{const}; \\ &Y_0: \quad I_4 = e^{\nu(x)}\dot{t}^2 - \dot{x}^2 - e^{\mu(x)}\left(\dot{y}^2 + \dot{z}^2\right) = \text{const}. \end{aligned} \tag{28}$$

It has been shown that in these Lie symmetries **X**0, **X**1, **X**2, and **X**3 are isometries and correspond to the onservation of energy, linear momentum in *y*-direction, linear momentum in *z*-direction, and angular momentum ,and**Y**0 is the symmetry corresponding to the Lagrange equation. It is important to note that all the isometries are independent of parameter *s*.

IV. Lie symmetry classification of a plane symmetry static space-time

4.1Theplane symmetry static space-time with sixLie symmetries

From the five basic Lie symmetry groups of a plane symmetry static space-time, it can be observed that the metric parameters $\nu(x)$ and $\mu(x)$ can take any value; therefore, there may be infinitely many classes for five Lie symmetries. It can be seen that a plane symmetry static space-time consists of an infinite number of classifications. However, we provide some examples of metrics with six Lie symmetries of a plane symmetry static space-time in section.
For the $\nu(x)$ and $\mu(x)$ of the plane symmetry static space-time with six Lie symmetries, we can give the following six forms:

a. If the metric parameters $\nu(x) = \frac{x}{a}, \mu(x) = \frac{x}{b}; \ a \neq b$, five symmetries **X**0, **X**1, **X**2, **X**3, and **Y**0 are the same as given by Eqs.(21)-(25), andwe take sixth symmetry as

$$\eta^0 = -\frac{t}{2a}, \eta^1 = 1, \eta^2 = -\frac{y}{zb}, \eta^3 = -\frac{z}{2b}, \xi = 0, \ X_4 = -\frac{t}{2a}\frac{\partial}{\partial t} + \frac{\partial}{\partial x} - \frac{y}{2b}\frac{\partial}{\partial y} - \frac{z}{2b}\frac{\partial}{\partial z}. \tag{29}$$

The sixth conserved quantity corresponding tothis system is written as

$$I_6 = \frac{t\dot{t}}{a}e^{\frac{x}{a}} - 2\dot{x} - \frac{e^{x/b}}{b}\left(y\dot{y} - z\dot{z}\right) = \text{const} \tag{30}$$

b. If the metric parameters $\nu(x) = 0, \mu(x) = 2\ln\cosh^2\frac{x}{a}, \ a \neq 0$; The symmetries **X**0, **X**1, **X**2, **X**3, and **Y**0 of thesystem are the same as given in Eqs.(21)-(25),and the sixth symmetry along with the gauge function are

$$\eta^0 = 0, \xi = \eta^1 = \eta^2 = \eta^3 = 0; Y_1 = s\frac{\partial}{\partial t}. \tag{31}$$

This system possesses the sixth conserved quantity as

$$I_6 = 2\left(t - s\dot{t}\right) = \text{const} \tag{32}$$

c. If the metric parameters $\nu(x) = 0, \mu(x) = 2\ln\cos^2\frac{x}{a}, \ a \neq 0$; The symmetries **X**0, **X**1, **X**2, **X**3, and **Y**0 of thesystem are the same as given in Eqs.(21)-(25) and the sixth symmetry along with the gauge function and the sixth conserved quantity also are Eqs.(30) and (31)

d. If the metric parameters $\nu(x)=2\ln\left(\frac{x}{a}\right)^2,\ \mu(x)=2\ln\left(\frac{x}{b}\right)^\alpha,\quad \alpha\neq 0,\alpha\neq 2;$, and we take the sixth symmetryas

$$\begin{aligned}&\xi=s,\eta^1=\frac{x}{2},\eta^2=\frac{2-\alpha}{4}y,\eta^3=\frac{2-\alpha}{4}z,\eta^0=0;\\ &Y_1=s\frac{\partial}{\partial s}+\frac{x}{2}\frac{\partial}{\partial x}+\frac{2-\alpha}{4}y\frac{\partial}{\partial y}+\frac{2-\alpha}{4}z\frac{\partial}{\partial z}.\end{aligned}\tag{33}$$

This system possesses the sixth conserved quantity

$$I_6=s\left[\left(\frac{x}{a}\right)^2\dot{t}^2-\dot{x}^2-\left(\frac{x}{b}\right)^\alpha\left(\dot{y}^2+\dot{z}^2\right)\right]+x\dot{x}+\frac{2-\alpha}{2}\left(\frac{x}{b}\right)^\alpha\left(y\dot{y}+z\dot{z}\right)=\text{const}\tag{34}$$

e. If the metric parameters $\nu(x)=2\ln\left(\frac{x}{b}\right)^\alpha,\ \mu(x)=2\ln\left(\frac{x}{a}\right)^2,\quad \alpha\neq 0,\alpha\neq 2$,the metric admits minimal set of Lie symmetries and following symmetry (homothety)

$$Y_1=s\frac{\partial}{\partial s}+\frac{2-\alpha}{4}t\frac{\partial}{\partial t}+\frac{x}{2}\frac{\partial}{\partial x}.\tag{35}$$

The plane symmetry static space-time possesses the sixth conserved quantity

$$I_6=s\left[\left(\frac{x}{b}\right)^2\dot{t}^2-\dot{x}^2-\left(\frac{x}{a}\right)^\alpha\left(\dot{y}^2+\dot{z}^2\right)\right]+x\dot{x}+\frac{2-\alpha}{2}\left(\frac{x}{a}\right)^\alpha t\dot{t}=\text{const}\tag{36}$$

f. If the metric parameters $\nu(x)=2\ln\left(\frac{x}{a}\right)^\beta,\ \mu(x)=2\ln\left(\frac{x}{b}\right)^\alpha,\quad 2\neq\alpha\neq\beta\neq 2$,which it admits a scaling symmetry (homothety)

$$Y_1=s\frac{\partial}{\partial s}+\frac{2-\beta}{4}t\frac{\partial}{\partial t}+\frac{x}{2}\frac{\partial}{\partial x}+\frac{2-\alpha}{4}y\frac{\partial}{\partial y}+\frac{2-\alpha}{4}z\frac{\partial}{\partial z},\tag{37}$$

In addition to the five symmetries given in Eqs.(20)-(24). This system possesses the sixth conserved quantity

$$I_6=s\left[\left(\frac{x}{a}\right)^\beta\dot{t}^2-\dot{x}^2-\left(\frac{x}{b}\right)^\alpha\left(\dot{y}^2+\dot{z}^2\right)\right]+x\dot{x}+\frac{2-\beta}{2}\left(\frac{x}{a}\right)^\beta t\dot{t}+\frac{2-\alpha}{2}\left(\frac{x}{b}\right)^\alpha=\text{const}\tag{38}$$

4.2 **Theplane symmetry static space-time with seven Lie symmetries**

In this section, the classes for seven Lie symmetries are given. There are two classes ofthe plane symmetry static space-time that admit seven Lie symmetries

Class 1 if $\nu(x)=2\ln\cosh^2\frac{x}{a},\mu(x)=2\ln\cosh^2\frac{x}{a}$, the Lagrange equations of plane symmetry static space-time are driven as

$$\begin{aligned}&\ddot{t}=-\frac{4}{a}\frac{1}{\cosh\frac{x}{a}}\left(1-\cosh^2\frac{x}{a}\right)\dot{x}\dot{t}=\alpha_0,\\ &\ddot{x}=\frac{4}{a}\cosh\frac{x}{a}\left(1-\cosh^2\frac{x}{a}\right)\dot{t}^2-\frac{4}{a}\cosh\frac{x}{a}\left(1-\cosh^2\frac{x}{a}\right)\left(\dot{y}^2+\dot{z}^2\right)=\alpha_1,\\ &\ddot{y}=-\frac{4}{a}\frac{1}{\cosh\frac{x}{a}}\left(1-\cosh^2\frac{x}{a}\right)\dot{x}\dot{y}=\alpha_2,\ddot{z}=-\frac{4}{a}\frac{1}{\cosh\frac{x}{a}}\left(1-\cosh^2\frac{x}{a}\right)\dot{x}\dot{z}=\alpha_3.\end{aligned}\tag{39}$$

Equations (39) admits seven Lie symmetries, four of which are given in Eqs.(21)-(25) and the Eqs.(17)~(20) which also possess two additional Lie symmetries in the following forms:

$$X_5=y\frac{\partial}{\partial t}+t\frac{\partial}{\partial y},\ X_6=z\frac{\partial}{\partial t}+t\frac{\partial}{\partial z}.\tag{40}$$

Conserved quantities corresponding to these symmetries are given as

$$I_5 = 2\ln\cosh^2\frac{x}{a}\left(t\dot{y} - y\dot{t}\right) = \text{const}, I_6 = 2\ln\cosh^2\frac{x}{a}\left(t\dot{z} - z\dot{t}\right) = \text{const}. \quad (41)$$

Class 2 if $\nu(x) = 2\ln\cos^2\frac{x}{a}, \mu(x) = 2\ln\cos^2\frac{x}{a}$, the Lagrange equations of plane symmetry static space-time are driven as

$$\begin{aligned} &\ddot{t} = \frac{4}{a}\tan\frac{x}{a}\dot{x}\dot{t} = \alpha_0, \ddot{x} = -\frac{2}{a}\sin\frac{2x}{a}\dot{t}^2 + \frac{2}{a}\sin\frac{2x}{a}\left(\dot{y}^2 + \dot{z}^2\right) = \alpha_1, \\ &\ddot{y} = \frac{4}{a}\tan\frac{x}{a}\dot{x}\dot{y} = \alpha_2, \ddot{z} = \frac{4}{a}\tan\frac{x}{a}\dot{x}\dot{z} = \alpha_3 \end{aligned} \quad (42)$$

Equations (42) admits seven Lie symmetries, four of which are given in Eqs.(21)-(25)and the Eqs.(17)-(20) which also possess two additional Lie symmetries in the following forms:

$$X_5 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, X_6 = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}. \quad (43)$$

Conserved quantities corresponding to these symmetries are given as

$$I_5 = 2\ln\cos^2\frac{x}{a}\left(t\dot{y} - y\dot{t}\right) = \text{const}, I_6 = 2\ln\cos^2\frac{x}{a}\left(t\dot{z} - z\dot{t}\right) = \text{const}. \quad (44)$$

Class 3 if $\nu(x) = 0, \mu(x) = 2\ln\left(\frac{x}{b}\right)^\alpha, 2 \neq \alpha \neq 0$, the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = 0 = \alpha_0, \ddot{x} = -\frac{2\alpha}{b}\left(\frac{x}{b}\right)^{\alpha-1}\left(\dot{y}^2 + \dot{z}^2\right) = \alpha_1, \ddot{y} = -\frac{2\alpha}{x}\dot{x}\dot{y} = \alpha_2, \ddot{z} = -\frac{2\alpha}{x}\dot{x}\dot{z} = \alpha_3. \quad (45)$$

Equations (45) admits seven Lie symmetries in clouding five basic Lie symmetries which are given in Eqs.(21)-(25) and two additional Lie symmetries are given as

$$Y_1 = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{2-\alpha}{4}y\frac{\partial}{\partial y} + \frac{2-\alpha}{4}z\frac{\partial}{\partial z}, Y_2 = s\frac{\partial}{\partial s}, G_2 = 2t. \quad (46)$$

Conserved quantities corresponding to two symmetries are given as

$$\begin{aligned} &I_5 = s\left[\left(\frac{x}{b}\right)^\alpha \dot{t}^2 - \dot{x}^2 - \left(\frac{x}{b}\right)^\alpha\left(\dot{y}^2 + \dot{z}^2\right)\right] + x\dot{x} - t\dot{t} + \frac{2-\alpha}{2}\left(\frac{x}{b}\right)^\alpha\left(y\dot{y} + z\dot{z}\right) = \text{const}, \\ &I_6 = t - s\dot{t} = \text{const}. \end{aligned} \quad (47)$$

Class 4 if $\nu(x)$ is arbitrary, and $\nu(x) \neq a\ln\frac{x}{b}, a \neq 0, \nu''(x) \neq 0, \mu(x) = 0,$ the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = -\frac{d\nu}{dx}\dot{x}\dot{t} = \alpha_0, \ddot{x} = \frac{1}{2}e^{\nu(x)}\frac{d\nu}{dx}\dot{t}^2 = \alpha_1, \ddot{y} = 0 = \alpha_2, \ddot{z} = 0 = \alpha_3, \quad (48)$$

Equations (48) admits the five basic Lie symmetries which are given in Eqs.(21)-(25) and two additional Lie symmetries are given as

$$Y_1 = s\frac{\partial}{\partial y}, G_1 = -2y, Y_2 = s\frac{\partial}{\partial z}, G_2 = -2z. \quad (49)$$

Conserved quantities corresponding to two additional Liesymmetries are

$$I_5 = s\dot{y} - y = \text{const}, \quad I_6 = s\dot{z} - z = \text{const}. \quad (50)$$

4.3 **Theplane symmetry static space-time with eight Lie symmetries**
In this section, the classes of the eight Lie symmetries presented. There are three classes of plane symmetry static space-time that admit eight Lie symmetries

Class 1 Ifthe metric parameters $\nu(x) = 2\ln\left(\frac{x}{b}\right)^\alpha, \mu(x) = 2\ln\left(\frac{x}{b}\right)^\alpha$, the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t}=-\frac{2\alpha}{x}\dot{x}\dot{t}=\alpha_0,\ddot{x}=\frac{2\alpha}{b}\left(\frac{x}{b}\right)^{\alpha-1}\dot{t}^2-\frac{2\alpha}{b}\left(\frac{x}{b}\right)^{\alpha-1}\left(\dot{y}^2+\dot{z}^2\right)=\alpha_1,$$
$$\ddot{y}=-\frac{2\alpha}{x}\dot{x}\dot{y}=\alpha_2,\ddot{z}=-\frac{2\alpha}{x}\dot{x}\dot{z}=\alpha_3. \quad (51)$$

This system possess eight Lie symmetries which including five basic forms of Lie symmetries are given by Eqs.(21)-(25)and three additional Lie symmetries are

$$X_5=y\frac{\partial}{\partial t}+t\frac{\partial}{\partial y},\ X_6=z\frac{\partial}{\partial t}+t\frac{\partial}{\partial z},$$
$$Y_1=s\frac{\partial}{\partial s}+\frac{2-\alpha}{4}t\frac{\partial}{\partial t}+\frac{x}{2}\frac{\partial}{\partial x}+\frac{2-\alpha}{4}y\frac{\partial}{\partial y}+\frac{2-\alpha}{4}z\frac{\partial}{\partial z}. \quad (52)$$

The plane symmetries static space-time possess the threeadditional conserved quantities

$$I_5=2\left(\frac{x}{b}\right)^{\alpha}\left(\dot{y}t-y\dot{t}\right)=\text{const},\quad I_6=2\left(\frac{x}{b}\right)^{\alpha}\left(\dot{z}t-z\dot{t}\right)=\text{const},$$
$$I_7=s\left[\left(\frac{x}{b}\right)^{\alpha}\dot{t}^2-\dot{x}^2-\left(\frac{x}{b}\right)^{\alpha}\left(\dot{y}^2+\dot{z}^2\right)\right]-\frac{2-\alpha}{2}\left(\frac{x}{b}\right)^{\alpha}t\dot{t}+x\dot{x} \quad (53)$$
$$+\frac{2-\alpha}{2}\left(\frac{x}{b}\right)^{\alpha}\left(y\dot{y}+z\dot{z}\right)=\text{const}.$$

Class 2 Ifthe metric parameters $\nu(x)=0,\mu(x)=2\ln\left(\frac{x}{a}\right)^2$, the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t}=0=\alpha_0,\ddot{x}=-\frac{4}{a^2}x\left(\dot{y}^2+\dot{z}^2\right)=\alpha_1,\ddot{y}=-\frac{4}{x}\dot{x}\dot{y}=\alpha_2,\ddot{z}=-\frac{4}{x}\dot{x}\dot{z}=\alpha_3 \quad (54)$$

This system possess eight Lie symmetries which including five basic forms of Lie symmetries are given by Eqs.(21)-(25)and three additional Lie symmetries are given as

$$Y_1=s\frac{\partial}{\partial t},Y_2=s\frac{\partial}{\partial s}+\frac{t}{2}\frac{\partial}{\partial t}+\frac{x}{2}\frac{\partial}{\partial x},Y_3=s^2\frac{\partial}{\partial s}+st\frac{\partial}{\partial t}+sx\frac{\partial}{\partial x}. \quad (55)$$

The plane symmetry static space-time possesses the threeadditional conserved quantities as

$$I_5=2\left(\frac{x}{b}\right)^{\alpha}\left(\dot{y}t-y\dot{t}\right)=\text{const},\quad I_6=2\left(\frac{x}{b}\right)^{\alpha}\left(\dot{z}t-z\dot{t}\right)=\text{const},$$
$$I_7=s\left[\left(\frac{x}{b}\right)^{\alpha}\dot{t}^2-\dot{x}^2-\left(\frac{x}{b}\right)^{\alpha}\left(\dot{y}^2+\dot{z}^2\right)\right]-\frac{2-\alpha}{2}\left(\frac{x}{b}\right)^{\alpha}t\dot{t}+x\dot{x} \quad (56)$$
$$+\frac{2-\alpha}{2}\left(\frac{x}{b}\right)^{\alpha}\left(y\dot{y}+z\dot{z}\right)=\text{const}.$$

Class 3 Ifthe metric parameters $\nu(x)=2\ln\left(\frac{x}{b}\right)^{\alpha},\mu(x)=0,$ the Lagrange equations of the plane symmetry static space-time are driven as

$$\ddot{t}=-\frac{2\alpha}{x}\dot{x}\dot{t}=\alpha_0,\ddot{x}=\frac{2\alpha}{b}\left(\frac{x}{b}\right)^{\alpha-1}\dot{t}^2=\alpha_1,\ddot{y}=0=\alpha_2,\ddot{z}=0=\alpha_3.,\ (57)$$

This system possesses eight Lie symmetries including five basic forms, which are given by Eqs.(21)-(25)and the three additional Lie symmetries are

$$Y_1 = s\frac{\partial}{\partial s} + \frac{2-\alpha}{4}t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}, Y_2 = s\frac{\partial}{\partial y}, G_2 = -2y, G_3 = -2z, Y_3 = s\frac{\partial}{\partial z}. \quad (58)$$

The plane symmetries static space-time possesses the threeadditional conserved quantities respectively as

$$I_5 = s\left[\left(\frac{x}{b}\right)^{\alpha}\dot{t}^2 - \dot{x}^2 - \left(\dot{y}^2 + \dot{z}^2\right)\right] + x\dot{x} - \frac{2-\alpha}{2}\left(\frac{x}{b}\right)^{\alpha} t\dot{t} + \left(y\dot{y} + z\dot{z}\right) = \text{const}, \quad (58)$$

$$I_6 = y - s\dot{y} = \text{const}, I_7 = z - s\dot{z} = \text{const}.$$

4.4**Theplane symmetry static space-time with nine Lie symmetries**

In this section, the classes for nine Lie symmetries are given. There are five classes of the plane symmetry static space-time that admit nine Lie symmetries

Class 1 Ifthe metric parameters $\nu(x) = 0, \mu(x) = \frac{x}{a},$ the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = 0 = \alpha_0, \ddot{x} = -\frac{1}{a}e^{\frac{x}{a}}\left(\dot{y}^2 + \dot{z}^2\right) = \alpha_1, \ddot{y} = -\frac{1}{a}\dot{x}\dot{y} = \alpha_2, \ddot{z} = -\frac{1}{a}\dot{x}\dot{z} = \alpha_3, \quad (59)$$

This system possesses nine Lie symmetries which including five basic forms of Lie symmetries are given by Eqs.(21)-(25)and four additional Lie symmetries are writtenas

$$X_5 = \frac{\partial}{\partial x} - \frac{y}{2a}\frac{\partial}{\partial y} - \frac{z}{2a}\frac{\partial}{\partial z}, X_7 = y\frac{\partial}{\partial x} + \left(-\frac{y^2}{4a} + \frac{z^2}{4a} + ae^{-\frac{x}{a}}\right)\frac{\partial}{\partial y} - \frac{yz}{2a}\frac{\partial}{\partial z},$$

$$X_8 = z\frac{\partial}{\partial x} - \left(\frac{y^2}{4a} - \frac{z^2}{4a} + ae^{-\frac{x}{a}}\right)\frac{\partial}{\partial z} - \frac{yz}{2a}\frac{\partial}{\partial y}, Y_1 = s\frac{\partial}{\partial y}, G_1 = 2t. \quad (60)$$

The plane symmetries static space-time possesses the fouradditional conserved quantities respectively as

$$I_5 = 2\dot{x} - \frac{e^{x/a}}{a}\left(y\dot{y} - z\dot{z}\right) = \text{const};$$

$$I_6 = 2\dot{x}y + \frac{e^{x/a}}{2a}\left[\dot{y}\left(z^2 - y^2 + 4a^2e^{-x/a}\right) - yz\dot{z}\right] = \text{const}; \quad (61)$$

$$I_7 = 2\dot{x}z + \frac{e^{x/a}}{2a}\left[\dot{z}\left(-z^2 + y^2 + 4a^2e^{-x/a}\right) - yz\dot{y}\right] = \text{const}; I_8 = t - s\dot{t} = \text{const}.$$

Class 2 Ifthe metric parameters $\nu(x) = \frac{x}{a}, \mu(x) = 0,$ the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = -\frac{1}{a}\dot{x}\dot{t} = \alpha_0, \ddot{x} = \frac{1}{a}e^{x/a}\dot{t}^2 = \alpha_1, \ddot{y} = 0 = \alpha_2, \ddot{z} = 0 = \alpha_3. \quad (62)$$

which admits the following four symmetries, in which **X5**and **X6**are isometries, along with the minimal set

$$X_5 = \frac{\partial}{\partial x} - \frac{t}{2a}\frac{\partial}{\partial t}, X_6 = t\frac{\partial}{\partial x} - \left(\frac{t^2}{4a} + ae^{x/a}\right)\frac{\partial}{\partial t},$$

$$Y_1 = s\frac{\partial}{\partial y}, G_1 = -2y, Y_2 = s\frac{\partial}{\partial z}, G_2 = -2z. \quad (63)$$

This plane symmetry static space-time possesses the fouradditional conserved quantities are

$$I_5 = \frac{t\dot{t}e^{x/a}}{a} - 2\dot{x} = \text{const}; I_6 = 2\dot{x}t + \left(t^2e^{x/a} + 4a^2\right)\frac{\dot{t}}{2a} = \text{const}; \quad (64)$$

$$I_7 = s\dot{y} - y = \text{const}; I_8 = s\dot{z} - z = \text{const}.$$

Class 3 Ifthe metric parameters $\nu(x) = 2\ln\cosh^2\frac{x}{a}, \mu(x) = 0,$ the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = -\frac{4}{a}\tanh\frac{x}{a}\dot{x}\dot{t} = \alpha_0; \ddot{x} = \frac{2}{a}\left(1 + 2\cosh^2\frac{x}{a}\right)\dot{t}^2 = \alpha_1; \qquad (65)$$
$$\ddot{y} = 0 = \alpha_2; \ddot{z} = 0 = \alpha_3.$$

For this metric the plane symmetry static space-time possesses the four additional symmetries are given by

$$X_5 = -\tanh\frac{x}{a}\sin\frac{t}{a}\frac{\partial}{\partial t} + \cos\frac{t}{a}\frac{\partial}{\partial x}; X_6 = \tanh\frac{x}{a}\cos\frac{t}{a} + \sin\frac{t}{a}\frac{\partial}{\partial x};$$
$$Y_1 = s\frac{\partial}{\partial y}, G_1 = -2y, Y_2 = s\frac{\partial}{\partial z}, G_2 = -2z. \qquad (66)$$

This plane symmetry static space-time possesses the fouradditional conserved quantities are given respectively as

$$I_5 = \dot{t}\sinh\frac{x}{a}\sin\frac{t}{a}\cosh\frac{x}{a} + \dot{x}\cos\frac{t}{a} = \text{const}; I_7 = s\dot{y} - y = \text{const};$$
$$I_6 = -\dot{t}\sinh\frac{x}{a}\cos\frac{t}{a}\cosh\frac{x}{a} + \dot{x}\sin\frac{t}{a} = \text{const}; I_8 = s\dot{z} - z = \text{const}. \qquad (67)$$

Class 4 Ifthe metric parameters $\nu(x) = 2\ln\cos^2\frac{x}{a}, \mu(x) = 0,$ the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = 4\tan\frac{x}{a}\dot{x}\dot{t} = \alpha_0, \ddot{x} = -\frac{2}{a}\sin\frac{2x}{a}\dot{t}^2 = \alpha_1, \ddot{y} = 0 = \alpha_2, \ddot{z} = 0 = \alpha_3. \qquad (68)$$

For this metric the plane symmetry static space-time possesses the four additional symmetries are given by

$$X_5 = -\tanh\frac{x}{a}\sin\frac{t}{a}\frac{\partial}{\partial t} + \cos\frac{t}{a}\frac{\partial}{\partial x}; X_6 = \tanh\frac{x}{a}\cos\frac{t}{a}\frac{\partial}{\partial t} + \sin\frac{t}{a}\frac{\partial}{\partial x};$$
$$Y_1 = s\frac{\partial}{\partial y}, G_1 = -2y, Y_2 = s\frac{\partial}{\partial z}, G_2 = -2z. \qquad (69)$$

This plane symmetry static space-time possesses the fouradditional conserved quantities are

$$I_5 = \dot{t}\sin\frac{x}{a}\sin\frac{t}{a}\cos\frac{x}{a} + \dot{x}\cos\frac{t}{a} = \text{const}; I_7 = s\dot{y} - y = \text{const};$$
$$I_6 = -\dot{t}\sin\frac{x}{a}\cos\frac{t}{a}\cos\frac{x}{a} + \dot{x}\sin\frac{t}{a} = \text{const}; I_8 = s\dot{z} - z = \text{const}. \qquad (70)$$

Class 5 Ifthe metric parameters $\nu(x) = 2\ln\left(\frac{x}{a}\right)^2, \mu(x) = 2\ln\left(\frac{x}{b}\right)^2,$ the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = -\frac{4}{x}\dot{x}\dot{t} = \alpha_0, \ddot{x} = \frac{4}{a^2}x\dot{t}^2 - \frac{4}{a^2}x\left(\dot{y}^2 + \dot{z}^2\right) = \alpha_1, \ddot{y} = -\frac{4}{x}\dot{x}\dot{y} = \alpha_2, \ddot{z} = -\frac{4}{x}\dot{x}\dot{z} = \alpha_3. (71)$$

The symmetries **X**0, **X**1, **X**2, **X**3, and **Y**0 of (71) are the same as given in Eqs.(21)-(25) and the four additional symmetries are given by

$$X_5 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}; X_6 = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z};$$
$$Y_1 = s\frac{\partial}{\partial s} + \frac{x}{2}\frac{\partial}{\partial x}; Y_2 = s^2\frac{\partial}{\partial s} + sx\frac{\partial}{\partial x}, G_2 = -2x^2. \qquad (72)$$

This plane symmetry static space-time possesses the fouradditional conserved quantities are written as

$$I_5 = 2\left(\frac{x}{b}\right)^2\left(\dot{y}t - y\dot{t}\right) = \text{const}; I_6 = 2\left(\frac{x}{b}\right)^2\left(\dot{z}t - z\dot{t}\right) = \text{const};$$
$$I_7 = s\left[\left(\frac{x}{b}\right)^2\dot{t}^2 - \dot{x}^2 - \left(\frac{x}{b}\right)^2\left(\dot{y}^2 + \dot{z}^2\right)\right] + x\dot{x} = \text{const}; \quad (73)$$
$$I_8 = s^2\left[\left(\frac{x}{b}\right)^2\dot{t}^2 - \dot{x}^2 - \left(\frac{x}{b}\right)^2\left(\dot{y}^2 + \dot{z}^2\right)\right] + 2sx\dot{x} - 2x^2 = \text{const}.$$

4.5 **Theplane symmetry static space-time with eleven Lie symmetries**

In this section, the classes for eleven Lie symmetries are given.

If the metric parameters $\nu(x) = \frac{x}{a}, \mu(x) = \frac{x}{a},$ the Lagrange equations of plane symmetry static space-time are driven by

$$\ddot{t} = -\frac{1}{a}\dot{x}\dot{t} = \alpha_0, \ddot{x} = \frac{1}{a}e^{\frac{x}{a}}\dot{t}^2 - \frac{1}{a}e^{\frac{x}{a}}\left(\dot{y}^2 + \dot{z}^2\right) = \alpha_1, \ddot{y} = -\frac{1}{a}\dot{x}\dot{y} = \alpha_2, \ddot{z} = -\frac{1}{a}\dot{x}\dot{z} = \alpha_3. \quad (74)$$

The symmetries **X**0, **X**1, **X**2, **X**3, and **Y**0 of (74) are the same as given in Eqs.(21)-(25) and the six additional symmetries are given by

$$X_5 = -\frac{t}{2a}\frac{\partial}{\partial t} + \frac{\partial}{\partial x} - \frac{y}{2a}\frac{\partial}{\partial y} - \frac{z}{2a}\frac{\partial}{\partial z}t; X_6 = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}; X_7 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y};$$
$$X_8 = y\frac{\partial}{\partial x} - \frac{yt}{2a}\frac{\partial}{\partial t} - \frac{yz}{2a}\frac{\partial}{\partial z} - \left(\frac{t^2}{4a} + \frac{y^2}{4a} - \frac{z^2}{4a} - ae^{-\frac{x}{a}}\right)\frac{\partial}{\partial y};$$
$$X_9 = z\frac{\partial}{\partial x} - \frac{zt}{2a}\frac{\partial}{\partial t} - \frac{zy}{2a}\frac{\partial}{\partial z} - \left(\frac{t^2}{4a} - \frac{y^2}{4a} + \frac{z^2}{4a} - ae^{-\frac{x}{a}}\right)\frac{\partial}{\partial z}; \quad (75)$$
$$X_{10} = t\frac{\partial}{\partial x} - \frac{ty}{2a}\frac{\partial}{\partial y} - \frac{tz}{2a}\frac{\partial}{\partial z} - \left(\frac{t^2}{4a} + \frac{y^2}{4a} + \frac{z^2}{4a} + ae^{-\frac{x}{a}}\right)\frac{\partial}{\partial t}.$$

The sixadditional conserved quantities of the system are

$$I_5 = 2\dot{x} + \frac{e^{x/a}}{a}\left(t\dot{t} - y\dot{y} - z\dot{z}\right) = \text{const}; I_6 = 2e^{\frac{x}{a}}\left(t\dot{z} - z\dot{t}\right) = \text{const};$$
$$I_7 = 2e^{\frac{x}{a}}\left(t\dot{y} - y\dot{t}\right) = \text{const};$$
$$I_8 = 2\dot{x}y + \frac{e^{x/a}}{2a}\left[2yt\dot{t} - 2yz\dot{z} + \left(z^2 - y^2 - t^2 + 4a^2e^{\frac{-x}{a}}\right)\dot{y}\right] = \text{const}; \quad (76)$$
$$I_9 = 2\dot{x}z + \frac{e^{x/a}}{2a}\left[2zt\dot{t} - 2yz\dot{y} + \left(z^2 - y^2 - t^2 + 4a^2e^{\frac{-x}{a}}\right)\dot{z}\right] = \text{const};$$
$$I_{10} = 2\dot{x}t + \frac{e^{x/a}}{2a}\left[-2y\dot{y}t - 2z\dot{z}t + \left(z^2 + y^2 + t^2 + 4a^2e^{\frac{-x}{a}}\right)\dot{t}\right] = \text{const};$$

4.6 **Theplane symmetry static space-time with seventeen Lie symmetries**

If the metric parameters $\nu(x) = 0, \mu(x) = 0,$ theplane symmetry static space-time is a Minkowski space-time, the Lagrange equations of the system are driven by

$$\ddot{t} = 0 = \alpha_0, \ddot{x} = 0 = \alpha_1, \ddot{y} = 0 = \alpha_2, \ddot{z} = 0 = \alpha_3. \quad (77)$$

The symmetries **X**0, **X**1, **X**2, **X**3, and **Y**0 of Eqs. (77) are the same as those given in Eqs.(21)-(25) and the 12additional symmetries are written in the following forms:

$$X_5 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}; X_6 = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}; X_7 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}; X_8 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x};$$
$$X_9 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}; X_{10} = \frac{\partial}{\partial x}; Y_1 = s\frac{\partial}{\partial t}, G_1 = t;$$
$$Y_2 = 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}; Y_3 = -s\frac{\partial}{\partial y}, G_3 = y; Y_4 = -s\frac{\partial}{\partial z}, G_4 = z; \quad (78)$$
$$Y_5 = s^2\frac{\partial}{\partial s} + st\frac{\partial}{\partial t} + sx\frac{\partial}{\partial x} + sy\frac{\partial}{\partial y} + sz\frac{\partial}{\partial z}, G_5 = t^2 - x^2 - y^2 - z^2;$$
$$Y_6 = -s\frac{\partial}{\partial x}, G_6 = x,$$

where, **X**5, **X**6, **X**7, **X**8, **X**9, and **X**10 are isometries and **Y**2 is homothety. The twelveadditional conserved quantities of the system are given as

$$I_5 = t\dot{y} - y\dot{t} = \text{const}; I_6 = t\dot{z} - z\dot{t} = \text{const}; I_7 = t\dot{x} - x\dot{t} = \text{const};$$
$$I_8 = x\dot{y} - \dot{x}y = \text{const}; I_9 = x\dot{z} - \dot{x}z = \text{const}; I_{10} = \dot{x} = \text{const}; I_{11} = t - s\dot{t} = \text{const};$$
$$I_{12} = s\left(\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2\right) - t\dot{t} + x\dot{x} + y\dot{y} + z\dot{z} = \text{const};$$
$$I_{13} = s\dot{y} + y = \text{const}; I_{14} = s\dot{z} - zy = \text{const}; \quad (79)$$
$$I_{15} = s^2\left(\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2\right) + 2s\left(-t\dot{t} + x\dot{x} + y\dot{y} + z\dot{z}\right) + t^2 - x^2 - y^2 - z^2 = \text{const};$$
$$I_{16} = s\dot{x} + x = \text{const}.$$

V. Conclusion

This article uses the method of analytical mechanics to study the gravitational field problem in four-dimensional space-time, and provides several important results: firstly, by introducing four-dimensional space-time coordinates as generalized coordinates and taking curve coordinates as independent variables, the Lagrangian equations of the plan symmetry static space-time are established; the secondly is to introduce transformation of the Lie groups and corresponding vector fields related to curve coordinates and four-dimensional space-time; thirdly, based on the invariance of the Lagrangian equations of the static space-time of plane symmetry under the transformation of the Lie group, the Lie symmetry determination equations (13) - (16) and a series of symmetry killing equations (17)-(20) for the four-dimensional static space-time are given; fourthly proposed and proved the Lie symmetry theorem for the plane symmetry static space-yime, and provided the structural equation (26) for the existence of the gravitational fields and the form of conserved quantities (27).

We have classified four-dimensional plane symmetry static space-time using the Lagrange equations and Lie symmetry theorem of the plane symmetric gravitational field, and obtained some useful conclusions; 1. There are five basic Lie symmetries (21) - (25) and conserved quantitiesin (27) static four-dimensional space-time of the planar symmetry;2. When the metric coefficients take six different forms, the static four-dimensional space-time of the planar symmetry has five basic Lie symmetries and an additional symmetry and conservation quantity;3. When there are four different forms of metric coefficient regions, the planar symmetric four-dimensional space-time has five basic symmetries and two additional Lie symmetries and conserved quantities;4. When the metric coefficients take three different forms, a plane symmetric gravitational field not only has five basic Lie symmetries, but also three additional Lie symmetries and conserved quantities; 5. When the metric coefficients take five different forms, the plane symmetric gravitational field has nine Lie symmetries and conserved quantities, including five basic Lie symmetries and four additional Lie symmetries and conserved quantities; 6. There is only one case where a static space-time of the plane symmetry has 11 Lie symmetries and conserved quantities, including 6 additional Lie symmetries and conserved quantities; 7. If the metric coefficients are all set to zero, a planar symmetric static space-time has 12 additional Lie symmetries and conserved quantities, as well as 5 basic Lie symmetries and conserved quantities in (28).

It should be noted that there are two basic methods to solve the Lie symmetry of differential equations. Firstly, the characteristic equation is given using the Lie symmetry of the equation, and then the first integral of the system is obtained by integration, as shown in reference [27]. The second is to solve the generator from the Killing equation of Lie symmetry, and use the Lie symmetry theorem provide the conserved quantity of the system's existence. The first method can provide non Noether conservation quantities and Noether form conservation quantities for the existence of the system; the second method provides the Noether form conservation for the existence of the system. This article adopts the second method to study the Lie symmetry properties of plane symmetric static spacetime, and uses the Lie symmetry theorem to find the Noether type conserved quantity that exists in the system. The system is classified using the Lie symmetry method and a series of results are obtained. The conclusion presented in this article is consistent with existing findings.

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