The International Journal of Engineering and Science (IJES)
|| Volume || 12 || Issue || 6 || June || Pages || PP 48-53 || 2023 ||
ISSN (e): 2319-1813 ISSN (p): 20-24-1805

Mathematical Derivation of the Second order Kalman filter Based on Estimation and Prediction

Iskanderani, Ahmed I.
Department of Electrical and Computer Engineering,
King Abdulaziz University, Jeddah, Saudi Arabia

---***ABSTRACT***---
This paper presents the use of the innovations approach for estimating signals described by linear second-order vector difference equations. The approach results in a recursive one-stage prediction estimator in second-order form, which preserves the structure of the signal model with innovations feedback. It is shown that the second order Kalman Filter is a special case of this estimator. Furthermore, the innovations can be obtained through a recurrence relation based on the knowledge of one-stage prediction estimates and measurements. The paper also demonstrates that these results can be derived using an augmentation approach based on Kalman filtering results. Interestingly, the computational complexity of the first-order estimator equations and the second-order estimator equation is the same, indicating no difference in online computations between the two estimators. However, the second-order estimator is more elegant in terms of its mathematical structure as a second-order linear dynamical system and in terms of the decoupling of the prediction and filtered estimates. The findings in this paper contribute to the field of signal processing and estimation theory. In this paper we derive the equations for the second order Kalman Filter based on estimation and predictions in two equivalent derivations.

Keywords: *filtering, prediction, estimation, signal processing, Kalman filter, discrete state space models.*

Date of Submission: 02-06-2023 Date of Acceptance: 12-06-2023

I. INTRODUCTION

The Kalman filter is a widely used algorithm for state estimation in linear and nonlinear systems. It was developed by Rudolf Kalman in the early 1960s and has since been widely used in various fields, including aerospace, robotics, and finance. The basic idea of the Kalman filter is to estimate the state of a system by fusing noisy measurements with a mathematical model of the system.

The estimation of signals described by linear second-order vector difference equations has been an area of active research in recent years. One approach that has shown promise is the innovations approach, which has been successfully applied to such signals. Specifically, it has been demonstrated that the innovations approach can yield a recursive one-stage prediction estimator in second-order form that preserves the structure of the signal model with innovations feedback. The second order Kalman filter is a special case of this estimator, highlighting the potential of this approach. This finding has been supported by numerous studies (e.g., [2], [3], [4]).

The innovations approach has also been shown to be amenable to computation using a recurrence relation based on one-stage prediction estimates and measurements. This recurrence relation can be derived using an augmentation approach based on Kalman filtering results. These results have been explored and discussed in detail in the literature [5], [6].

Interestingly, despite the increased mathematical complexity of the second-order estimator, the computational complexity of the first order and second-order estimators is the same, meaning that there is no difference in online computations between the two estimators. However, the second-order estimator is more elegant in its mathematical structure as a second-order linear dynamical system, and the prediction and filtered estimates' decoupling is improved. This finding has been documented in studies that have compared the two estimators [7], [8]. In this paper we derive the second order Kalman Filter based on estimation and prediction using the innovations approach.

In conclusion, the second order Kalman Filter (SOKF) is a useful tool for state estimation in nonlinear systems, but it has limitations in terms of computational complexity. Higher-order Kalman filters can provide better performance than the SOKF in nonlinear systems and have been successfully applied in various engineering fields. Efficient implementation of high-order Kalman filters is also essential for real-time applications.

II. MATHEMATICAL DERIVATIONS

Based on the recent paper written by Iskanderani [1], the following equations were derived:

Consider the model for the signal being considered is expressed as a p'th-order linear VDE in the form of:

$$\boldsymbol{x_{k+1}} = \sum_{j=1}^{p} \boldsymbol{A_k^j x_{k-j+1}} + \boldsymbol{\Gamma_k \omega_k} \tag{1}$$

$$\boldsymbol{y_k} = \sum_{j=1}^{p} \boldsymbol{C_k^j x_{k-j+1}} + \boldsymbol{v_k} \tag{2}$$

The one-stage prediction estimator for equations (1) and (2) may be given by:

$$\boldsymbol{\hat{x}_{k+1|k}} = \sum_{j=1}^{p} \boldsymbol{A_k^j \hat{x}_{k-j+1|k-j}} + \sum_{i=1}^{p} \boldsymbol{G_k^i \tilde{y}_{k-i+1}} \ , \boldsymbol{k = 1, 2}, \ldots \tag{3}$$

with the initial condition

$$\boldsymbol{\hat{x}_{j|j-1}} = \boldsymbol{\bar{x}_j} \ , \ \boldsymbol{j = -p+2, -p+3}, \ldots, \boldsymbol{-1, 0, 1} \tag{4}$$

The innovations are given by

$$\boldsymbol{\tilde{y}_k} + \sum_{i=1}^{p-1} \boldsymbol{H_k^i \tilde{y}_{k-i}} = \boldsymbol{y_k} - \sum_{j=1}^{p} \boldsymbol{C_k^j \hat{x}_{k-j+1|k-j}} \ , \boldsymbol{k = p, p+1}, \ldots \tag{5}$$

with the initial conditions

$$\boldsymbol{\tilde{y}_m} = \boldsymbol{y_m} - \sum_{j=1}^{p} \boldsymbol{C_m^j \bar{x}_{m-j+1}} \ , \ m = 1,2, \ldots, p-1 \tag{6}$$

The $n \times m$ gain matrices are given by

$$\boldsymbol{G}_k^i = \sum_{m=1}^{i} \boldsymbol{A}_k^{p-m+1} E[\boldsymbol{x}_{k-p+m} \tilde{\boldsymbol{y}}_{k-p+i}^T] \boldsymbol{K}_{k-p+i}^{-1} \tag{7}$$

$$\boldsymbol{K}_k = \sum_{j=1}^{p} \sum_{i=1}^{p} C_k^j E[\tilde{\boldsymbol{x}}_{k-j+1|k-1} \tilde{\boldsymbol{x}}_{k-i+1|k-1}] C_k^{iT} + \boldsymbol{R}_k \tag{8}$$

where $i = 1,2, \ldots, p$, and

$$\boldsymbol{H}_k^i = \sum_{m=i+1}^{p} C_k^m E[\boldsymbol{x}_{k-m+1} \tilde{\boldsymbol{y}}_{k-i}^T] E[\tilde{\boldsymbol{y}}_{k-i} \tilde{\boldsymbol{y}}_{k-i}^T] \boldsymbol{K}_{k-i}^{-1} \tag{9}$$

III. Kalman Filter as a Special Case of Estimation and Prediction

If $p = 1$ the signal model (1) and (2) reduces to the conventional first- order state-variable model

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_k x_k + \boldsymbol{\Gamma}_k \boldsymbol{\omega}_k \tag{10}$$

$$\mathbf{y}_k = \boldsymbol{C}_k \boldsymbol{x}_k + \boldsymbol{v}_k \tag{11}$$

Which is the well known one-stage prediction estimator (Kalman filter). Hence, the high order Kalman Filter is represented by the equations 1-4 above.

If $p = 2$ the signal model (1) and (2) reduces to the conventional second order discrete state-variable model and the second order Kalman Filter may be shown to follow the following equations:

$$\boldsymbol{y}_{k+1} \quad = \boldsymbol{A}_k^1 \boldsymbol{x}_k + \boldsymbol{A}_k^2 \boldsymbol{x}_{k-1} + \boldsymbol{\Gamma}_k \boldsymbol{\omega}_k \tag{12}$$

$$\boldsymbol{y}_k \quad = \boldsymbol{C}_k^1 \boldsymbol{x}_k + \boldsymbol{C}_{k-1}^2 + \boldsymbol{v}_k \tag{13}$$

The one-stage prediction estimator for the system (3) and (4), with the stated assumptions, is of the form:

$$\hat{\boldsymbol{x}}_{k+1|k} = \boldsymbol{A}_k \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{D}_k \hat{\boldsymbol{x}}_{k-1|k-2} + \boldsymbol{G}_k^2 \tilde{\boldsymbol{y}}_k + \boldsymbol{D}_k \boldsymbol{G}_{k-1}^1 \tilde{\boldsymbol{y}}_{k-1} \tag{14}$$

for $k = 1,2, \ldots$, with initial vectors

$$\hat{\boldsymbol{x}}_{0|-1} = \bar{\boldsymbol{x}}_0 \text{ and } \hat{\boldsymbol{x}}_{1|0} = \bar{\boldsymbol{x}}_1 \tag{15}$$

The innovations satisfy:

$$\tilde{\boldsymbol{y}}_k + \boldsymbol{E}_k \boldsymbol{G}_{k-1}^1 \tilde{\boldsymbol{y}}_{k-1} = \boldsymbol{y}_k - \boldsymbol{C}_k \hat{\boldsymbol{x}}_{k|k-1} - \boldsymbol{E}_k \hat{\boldsymbol{x}}_{k-1|k-2}, \ k = 2,3, \ldots \tag{16}$$

with the initial condition

$$\tilde{\boldsymbol{y}}_1 = \boldsymbol{y}_1 - \boldsymbol{C}_1 \bar{\boldsymbol{x}}_1 - \boldsymbol{E}_1 \bar{\boldsymbol{x}}_0 \tag{8}$$

The gains $\boldsymbol{G}_k{}^1$ and $\boldsymbol{G}_k{}^2$ are given respectively by

$$\boldsymbol{G}_k^1 \quad = \left[\Sigma_{k|k-1} \boldsymbol{C}_k^T + \boldsymbol{\Pi}_{k|k-1}^T \quad \boldsymbol{E}_k^T\right] \boldsymbol{K}_k^{-1},$$

$$\boldsymbol{G}_k^2 \quad = \boldsymbol{A}_k \boldsymbol{G}_k^1 + \boldsymbol{D}_k \left[\boldsymbol{\Pi}_{k|k-1} \boldsymbol{C}_k^T + \Sigma_{k-1|k-1} \quad \boldsymbol{E}_k^T\right] \boldsymbol{K}_k^{-1} \tag{17}$$

Where the Kalman gain K_k can be shown to be:

$$\boldsymbol{K}_k = \quad \boldsymbol{C}_k \Sigma_{k|k-1} \boldsymbol{C}_k^T + \boldsymbol{C}_k \Pi_{k|k-1}^T \boldsymbol{E}_k^T + \boldsymbol{E}_k \boldsymbol{\Pi}_{k|k-1} \boldsymbol{C}_k^T + \boldsymbol{E}_k \Sigma_{k-1|k-1} \boldsymbol{E}_k^T + \boldsymbol{R}_k \tag{18}$$

The associated covariances are given by:

$$\Sigma_{k+1|k} = \left(\boldsymbol{D}_k - \boldsymbol{G}_k^2 E_k\right)\left(\Sigma_{k-1|k-1} \boldsymbol{D}_k^T + \boldsymbol{\Pi}_{k|k-1} \boldsymbol{A}_k^T\right)$$

$$+\left(\boldsymbol{A}_k - \boldsymbol{G}_k^2 \boldsymbol{C}_k\right)\left(\boldsymbol{\Pi}_{k|k-1}^T \boldsymbol{D}_k^T + \Sigma_{k|k-1} \boldsymbol{A}_k^T\right) + \boldsymbol{\Gamma}_k \boldsymbol{Q}_k \boldsymbol{\Gamma}_k^T \tag{19}$$

$$\Sigma_{k|k} = \Sigma_{k|k-1} - \boldsymbol{G}_k^1 \boldsymbol{C}_k \Sigma_{k|k-1} - \boldsymbol{G}_k^1 \boldsymbol{E}_k \boldsymbol{\Pi}_{k|k-1} \tag{20}$$

$$\boldsymbol{\Pi}_{k+1|k} = \Sigma_{k|k} \boldsymbol{A}_k^T + \left(\boldsymbol{\Pi}_{k|k-1}^T - \boldsymbol{G}_k^1 \boldsymbol{C}_k \boldsymbol{\Pi}_{k|k-1}^T - \boldsymbol{G}_k^1 \boldsymbol{E}_k \Sigma_{k-1|k-1}\right) \boldsymbol{D}_k^T \tag{21}$$

with initial matrices given by

$$\begin{aligned}\boldsymbol{\Sigma}_{0|0} &= E[(\boldsymbol{x}_0-\bar{\boldsymbol{x}}_0)(\boldsymbol{x}_0-\bar{\boldsymbol{x}}_0)^T],\\ \boldsymbol{\Sigma}_{1|1} &= E[(\boldsymbol{x}_1-\bar{x}_1)(\boldsymbol{x}_1-\bar{\boldsymbol{x}}_1)^T],\\ \boldsymbol{\Pi}_{1|0} &= E[(\boldsymbol{x}_0-\bar{\boldsymbol{x}}_0)(\boldsymbol{x}_1-\bar{\boldsymbol{x}}_1)^T],\end{aligned} \tag{22}$$

In addition, the filtered estimate is given by equation:

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{G}_k^1\tilde{\boldsymbol{y}}_k \tag{23}$$

Noting that the recursive formulas for the covariance are given by:

$$\begin{aligned}\boldsymbol{P}_{k|k-1} &= E\left\{\begin{bmatrix}\widetilde{\boldsymbol{x}}_{k-1|k-1}\\ \widetilde{\boldsymbol{x}}_{k|k-1}\end{bmatrix}\begin{bmatrix}\widetilde{\boldsymbol{x}}_{k-1|k-1}^T & \widetilde{\boldsymbol{x}}_{k|k-1}^T\end{bmatrix}\right\}\\ &= \begin{bmatrix}E\left[\widetilde{\boldsymbol{x}}_{k-1|k-1}\widetilde{\boldsymbol{x}}_{k-1|k-1}^T\right] & E\left[\widetilde{\boldsymbol{x}}_{k-1|k-1}\widetilde{\boldsymbol{x}}_{k|k-1}^T\right]\\ E\left[\widetilde{\boldsymbol{x}}_{k|k-1}\widetilde{\boldsymbol{x}}_{k-1|k-1}^T\right] & E\left[\widetilde{\boldsymbol{x}}_{k|k-1}\widetilde{\boldsymbol{x}}_{k|k-1}^T\right]\end{bmatrix}\\ &\triangleq \begin{bmatrix}\boldsymbol{\Sigma}_{k-1|k-1} & \boldsymbol{\Pi}_{k|k-1}\\ \boldsymbol{\Pi}_{k|k-1}^T & \boldsymbol{\Sigma}_{k|k-1}\end{bmatrix}\end{aligned} \tag{24}$$

and $\boldsymbol{G}_k{}^1$ and $\boldsymbol{G}_k^2$ can be found the following matrix equation:

$$\begin{aligned}\begin{bmatrix}\boldsymbol{G}_k^1\\ \boldsymbol{G}_k^2\end{bmatrix} = &\begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}\\ \boldsymbol{D}_k & \boldsymbol{A}_k\end{bmatrix}\begin{bmatrix}\boldsymbol{\Sigma}_{k-1|k-1} & \boldsymbol{\Pi}_{k|k-1}\\ \boldsymbol{\Pi}_{k|k-1}^T & \boldsymbol{\Sigma}_{k|k-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{E}_k^T\\ \boldsymbol{C}_k^T\end{bmatrix}\times\\ &\left[\begin{bmatrix}\boldsymbol{E}_k & \boldsymbol{C}_k\end{bmatrix}\begin{bmatrix}\boldsymbol{\Sigma}_{k-1|k-1} & \boldsymbol{\Pi}_{k|k-1}\\ \boldsymbol{\Pi}_{k|k-1}^T & \boldsymbol{\Sigma}_{k|k-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{E}_k^T\\ \boldsymbol{C}_k^T\end{bmatrix}+\boldsymbol{R}_k\right]^{-1}\end{aligned} \tag{25}$$

Which can be shown to be as follows:

$$\boldsymbol{G}_k^1 = \left(\boldsymbol{\Sigma}_{k-1|k-1}\boldsymbol{E}_k^T + \boldsymbol{\Pi}_{k|k-1}\boldsymbol{C}_k^T\right)\boldsymbol{K}_k^{-1} \tag{26}$$

$$\begin{aligned}\boldsymbol{G}_k^2 &= \left(\boldsymbol{D}_k\boldsymbol{\Sigma}_{k-1|k-1}\boldsymbol{E}_k^T + \boldsymbol{D}_k\boldsymbol{\Pi}_{k|k-1}\boldsymbol{C}_k^T\right.\\ &\quad\left.+\boldsymbol{A}_k\boldsymbol{\Pi}_{k|k-1}^T\boldsymbol{E}_k^T + \boldsymbol{A}_k\boldsymbol{\Sigma}_{k|k-1}\boldsymbol{C}_k^T\right)\boldsymbol{K}_k^{-1}\\ &= \left(\boldsymbol{D}_k\boldsymbol{\Sigma}_{k-1|k-1}\boldsymbol{E}_k^T + \boldsymbol{D}_k\boldsymbol{\Pi}_{k|k-1}\boldsymbol{C}_k^T\right)\boldsymbol{K}_k^{-1} + \boldsymbol{A}_k\boldsymbol{G}_k^1\\ &= \boldsymbol{A}_k\boldsymbol{G}_k^1 + \boldsymbol{D}_k\left(\boldsymbol{\Sigma}_{k-1|k-1}\boldsymbol{E}_k^T + \boldsymbol{\Pi}_{k|k-1}\boldsymbol{C}_k^T\right)\boldsymbol{K}_k^{-1}\end{aligned} \tag{27}$$

where the $m \times m$ $\boldsymbol{K}_k$ matrix is defined by:

$$\begin{aligned}\boldsymbol{K}_k \triangleq\ &\boldsymbol{C}_k\boldsymbol{\Sigma}_{k|k-1}\boldsymbol{C}_k^T + \boldsymbol{C}_k\boldsymbol{\Pi}_{k|k-1}^T\boldsymbol{E}_k^T + \boldsymbol{E}_k\boldsymbol{\Pi}_{k|k-1}\boldsymbol{C}_k^T\\ &+\boldsymbol{E}_k\boldsymbol{\Sigma}_{k-1|k-1}\boldsymbol{E}_k^T + \boldsymbol{R}_k\end{aligned} \tag{28}$$

Thus, the expressions for the gain matrices in equations (26) through (28) are the same as were presented earlier by equations (17) and (18). The one-stage prediction estimator for the system (3) and (4) is depicted in the figure below:

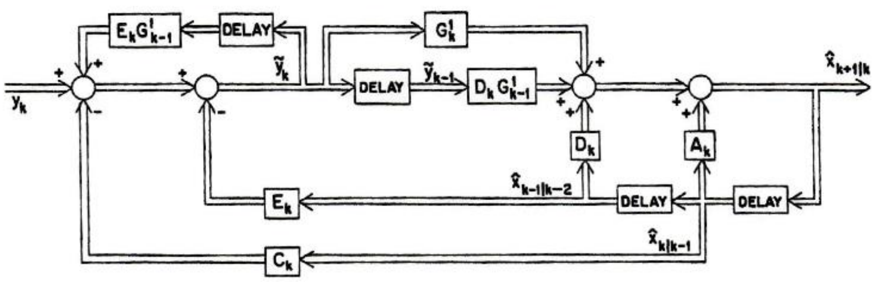

Fig. 1. The one-stage prediction estimator for the system (3) and (4)

Note that the block diagram in the figure maintains the second-order VDE signal model with feedback loops for innovations. Given the jointly Gaussian of $\{\boldsymbol{x}_k\}$ and $\{\boldsymbol{y}_k\}$, ensures that $\boldsymbol{x}_k$ conditioned on $\boldsymbol{y}_{k-1}, \hat{\boldsymbol{x}}_{k|k-1}$, is Gaussian. Likewise, $\hat{\boldsymbol{x}}_{k|k}$ and $\tilde{\boldsymbol{y}}_k$ are also Gaussian. The estimator takes $\{\boldsymbol{y}_k\}$ as input and produces $\{\hat{\boldsymbol{x}}_{k|k-1}\}$ as output in real time, with the specific values of output $\{\hat{\boldsymbol{x}}_{k|k-1}\}$ being dependent on the measurements $y_1, y_2, \cdots, y_{k-1}$. However, the gains $\boldsymbol{G}_k{}^1$ and $\boldsymbol{G}_k{}^2$ and the associated covariances $\boldsymbol{\Sigma}_{k|k-1}$, $\Sigma_{k|k}$ and $\boldsymbol{\Pi}_{k|k-1}$ are all independent of the measurements. For this reason, the gains and the associated covariances can be computed off-line before the estimator is actually run. Importantly, the estimator preserves the form of the second-order VDE signal model with feedback loops.

Alternatively, we can reconsider the system described by (3) and (4) with the stated assumptions mentioned by Iskanderani [1]. Hence, equations (3) and (4) can be represented in terms of a first order (state-variable) equivalent form. Let:

$$\begin{aligned} \boldsymbol{x}_k^1 &= \boldsymbol{x}_{k-1}, \\ \boldsymbol{x}_k^2 &= \boldsymbol{x}_k \end{aligned} \tag{29}$$

Then equations (3.3) and (3.4) can be put into the following first-order form:

$$\begin{aligned} \begin{bmatrix} \boldsymbol{x}_{k+1}^1 \\ \boldsymbol{x}_{k+1}^2 \end{bmatrix} &= \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ \boldsymbol{D}_k & \boldsymbol{A}_k \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_k^1 \\ \boldsymbol{x}_k^2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\Gamma}_k \end{bmatrix} \boldsymbol{w}_k, \\ \boldsymbol{y}_k &= [\boldsymbol{E}_k \quad \boldsymbol{C}_k] \begin{bmatrix} \boldsymbol{x}_k^1 \\ \boldsymbol{x}_k^2 \end{bmatrix} + \boldsymbol{v}_k, \end{aligned} \tag{30}$$

which can be written as:

$$\begin{aligned} \boldsymbol{z}_{k+1} &= \boldsymbol{\Phi}_k \boldsymbol{z}_k + \boldsymbol{\Lambda}_k \boldsymbol{w}_k, \\ \boldsymbol{y}_k &= \boldsymbol{\theta}_k \boldsymbol{z}_k + \boldsymbol{v}_k \end{aligned} \tag{31}$$

where the $2n$-vector z_k is defined by

$$\begin{aligned} \boldsymbol{z}_k &= \left[\boldsymbol{x}_k^1 \quad \boldsymbol{x}_k^2\right]^T \\ &= [\boldsymbol{x}_{k-1} \ \boldsymbol{x}_k]^T \end{aligned} \tag{32}$$

Φ_k is a $2n \times 2n$ matrix defined by

$$\boldsymbol{\Phi}_k \triangleq \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ \boldsymbol{D}_k & \boldsymbol{A}_k \end{bmatrix} \tag{33}$$

where I is the $n \times n$ identity matrix. Λ_k is a $2n \times n$ matrix defined by

$$\boldsymbol{\Lambda}_k \triangleq \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\Gamma}_k \end{bmatrix} \tag{34}$$

Θ_k is an $m \times 2n$ matrix defined by

$$\boldsymbol{\Theta}_k \triangleq [\boldsymbol{E}_k \quad \boldsymbol{C}_k] \tag{35}$$

The assumptions of Section 3 can be stated in terms of the new representation as follows:

1. The input noise $\{\boldsymbol{w}_k\}$ and the output noise $\{\boldsymbol{v}_k\}$ are independent and gaussian processes with means and covariances as given in Section 3.
2. The initial state $\boldsymbol{z}_0$ is a gaussian random vector with mean:

$$\bar{\boldsymbol{z}}_0 = E\begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_1 \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{x}}_0 \\ \bar{\boldsymbol{x}}_1 \end{bmatrix} \tag{36}$$

and covariance:

$$\begin{aligned} \boldsymbol{P}_{0|-1} &= E[(\boldsymbol{z}_0 - \bar{\boldsymbol{z}}_0)(\boldsymbol{z}_0 - \bar{\boldsymbol{z}}_0)^T] \\ &= E\left\{\begin{bmatrix} (\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_0) \\ (\boldsymbol{x}_1 - \bar{\boldsymbol{x}}_1) \end{bmatrix} [(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_0)^T \quad (\boldsymbol{x}_1 - \bar{\boldsymbol{x}}_1)^T]\right\} \\ &= \begin{bmatrix} E[(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_0)(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_0)^T] & E[(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_0)(\boldsymbol{x}_1 - \bar{\boldsymbol{x}}_1)^T] \\ E[(\boldsymbol{x}_1 - \bar{\boldsymbol{x}}_1)(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_0)^T] & E[(\boldsymbol{x}_1 - \bar{\boldsymbol{x}}_1)(\boldsymbol{x}_1 - \bar{\boldsymbol{x}}_1)^T] \end{bmatrix} \\ &\triangleq \begin{bmatrix} \boldsymbol{\Sigma}_{0|0} & \boldsymbol{\Pi}_{1|0} \\ \boldsymbol{\Pi}_{1|0}^T & \boldsymbol{\Sigma}_{1|0} \end{bmatrix}. \end{aligned} \tag{37}$$

Noting that the initial state $\boldsymbol{z}_0$ is independent of $\{\boldsymbol{v}_k\}$ and $\{\boldsymbol{w}_k\}$ and $\boldsymbol{R}_k$ is the covariance matrix of $\{\boldsymbol{v}_k\}$ $is\ and\ mxm$ positive definite matrix. Hence, we have reached the same equations.

IV. APPLICATION: DISPLACEMENT/VELOCITY ESTIMATORS FOR UNDAMPED ELASTIC SYSTEMS

In this section, we examine a Gaussian stochastic process that can be seen as the output process of an undamped elastic system. To simplify the equations, we employ a modal decomposition technique. Through an exact discretization scheme, we derive a first-order discrete-time model defined at the time instances of discrete measurements. By manipulating algebraically, we obtain second-order vector recursions expressed solely in terms of displacements and velocities.

We develop the one-stage prediction estimator for the discrete-time model using two different approaches. Firstly, we apply Kalman filtering techniques to the first-order discrete-time model. Secondly, we directly apply the theory established to the second-order recursion, obtaining the one-stage prediction estimator directly in second-order form. Although both approaches yield equivalent estimators, there are notable differences in terms of dimensionality and computational considerations. Particularly, second-order estimators exhibit a substantial reduction in the number of operations required as online estimators.

Consider a linear second-order undamped elastic system described by

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(t) + \boldsymbol{K}\boldsymbol{x}(t) = 0 \tag{38}$$

where $\{x(t), t \geq t_0\}$ is an n-vector stochastic displacement process and $\{\dot{x}(t), t \geq t_0\}$ is an n-vector stochastic velocity process. Based on usual assumptions for elastic systems, assume M and K are $n \times n$ symmetric and positive definite matrices. Throughout, consider discrete-time measurements of the form

$$\boldsymbol{y}_k = \boldsymbol{C}\dot{\boldsymbol{x}}(t_k) + \boldsymbol{v}_k \tag{39}$$

or

$$\boldsymbol{y}_k = E\boldsymbol{x}(t_k) + \boldsymbol{v}_k \tag{40}$$

where $\{y_k\}$ is an m-vector measurement process, $\boldsymbol{C}$ and $\boldsymbol{E}$ are $m \times n$ matrices and the fixed measurements times $\{t_k\}$ satisfy $t_k < t_{k+1},\ k = 0,1, \ldots$ The measurement noise $\{\boldsymbol{v}_k\}$ is an m-vector, zero mean gaussian white-noise process with covariance:

$$E\left[\boldsymbol{v}_k\boldsymbol{v}_l^T\right] = \boldsymbol{R}_k\delta_{kl} \tag{41}$$

Assume the following:

(1) The initial vectors $x(t_0)$ and $\dot{x}(t_0)$ are jointly gaussian with zero means and covariances:

$$\begin{aligned} E[\boldsymbol{x}(t_0)\boldsymbol{x}^T(t_0)] &= \boldsymbol{\Sigma}^{xx}(t_0) \\ E[\boldsymbol{x}(t_0)\dot{\boldsymbol{x}}^T(t_0)] &= \boldsymbol{\Pi}^{x\dot{x}}(t_0) \\ E[\dot{\boldsymbol{x}}(t_0)\dot{\boldsymbol{x}}^T(t_0)] &= \boldsymbol{\Sigma}^{\dot{x}\dot{x}}(t_0) \end{aligned} \tag{42}$$

(2) The initial vectors $\boldsymbol{x}(t_0)$ and $\dot{\boldsymbol{x}}(t_0)$ are independent of $\{\boldsymbol{v}_k\}$.

(3) $\boldsymbol{R}_k$ is an $m \times m$ positive-definite matrix for each k.

It is convenient to make a basis change to modal coordinates. Let $\boldsymbol{\Psi}$ be an $n \times n$ non-singular matrix and let $\boldsymbol{\Omega}^2$ be an $n \times n$ diagonal matrix defined by:

$$\boldsymbol{\Psi}^T\boldsymbol{M}\boldsymbol{\Psi} = \boldsymbol{I},\ \boldsymbol{\Psi}^T\boldsymbol{K}\boldsymbol{\Psi} = \boldsymbol{\Omega}^2 \tag{43}$$

where,

$$\boldsymbol{\Omega}^2 = \text{diag}(w_1^2, w_2^2, \ldots, w_n^2) \tag{44}$$

Next, consider the transformation to modal coordinates defined by: $\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{\eta}$

Then the system (4.1), (4.2) and (4.3) can be written as:

$$\ddot{\boldsymbol{\eta}}(t) + \boldsymbol{\Omega}^2\boldsymbol{\eta}(t) = 0, \tag{45}$$

with discrete-time measurements of the form

$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{\Psi}\dot{\boldsymbol{\eta}}(t_k) + \boldsymbol{v}_k \quad \text{, or} \tag{46}$$

$$\boldsymbol{y}_k = \boldsymbol{E}\boldsymbol{\Psi}\boldsymbol{\eta}(t_k) + \boldsymbol{v}_k \tag{47}$$

The solution to the second-order differential equation (4.8) is given by:

$$\begin{bmatrix} \boldsymbol{\eta}(t) \\ \dot{\boldsymbol{\eta}}(t) \end{bmatrix} = \begin{bmatrix} \cos\boldsymbol{\Omega}(t - t_0) & \boldsymbol{\Omega}^{-1}\sin\boldsymbol{\Omega}(t - t_0) \\ -\boldsymbol{\Omega}\sin\boldsymbol{\Omega}(t - t_0) & \cos\boldsymbol{\Omega}(t - t_0) \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}(t_0) \\ \dot{\boldsymbol{\eta}}(t_0) \end{bmatrix} \tag{48}$$

We might focus on the system at discrete-time instants, $t_k, k = 0,1, \ldots$, the resulting recursions for $\boldsymbol{x}(t_k) = x_k$ and $\dot{\boldsymbol{x}}(t_k) = \dot{\boldsymbol{x}}_k$ are easily obtained. From Equation (4.11) and $x = \Psi\eta$

$$\begin{bmatrix} \boldsymbol{x}_{k+1} \\ \dot{\boldsymbol{x}}_{k+1} \end{bmatrix} = \boldsymbol{\Phi}_k \begin{bmatrix} \boldsymbol{x}_k \\ \dot{\boldsymbol{x}}_k \end{bmatrix} \tag{49}$$

where the $2n \times 2n$ matrix $\boldsymbol{\Phi}_k$ is defined by

$$\boldsymbol{\Phi}_k = \begin{bmatrix} \boldsymbol{\Psi}\cos\boldsymbol{\Omega}(t_{k+1} - t_k)\boldsymbol{\Psi}^{-1} & \boldsymbol{\Psi}\boldsymbol{\Omega}^{-1}\sin\boldsymbol{\Omega}(t_{k+1} - t_k)\boldsymbol{\Psi}^{-1} \\ -\boldsymbol{\Psi}\Omega\sin\Omega(t_{k+1} - t_k)\boldsymbol{\Psi}^{-1} & \boldsymbol{\Psi}\cos\boldsymbol{\Omega}(t_{k+1} - t_k)\boldsymbol{\Psi}^{-1} \end{bmatrix} \tag{50}$$

The discrete-time measurements are of the displacement form.

$$\boldsymbol{y}_k = \boldsymbol{E}\boldsymbol{x}_k + \boldsymbol{v}_k \tag{51}$$

or of the velocity form

$$\boldsymbol{y}_k = \boldsymbol{C}\dot{\boldsymbol{x}}_k + \boldsymbol{v}_k \tag{52}$$

V. CONCLUSIONS

The application of the innovations approach to estimating signals governed by linear second-order vector difference equations has been demonstrated. This results in a recursive one-stage prediction estimator in second-order form that preserves the signal model's structure with innovations feedback, with the second order Kalman Filter representing a special case. Moreover, it has been shown that the innovations can be obtained through a recurrence relation using one-stage prediction estimates and measurements. These findings can also be derived using an augmentation approach based on Kalman filtering results.

Interestingly, the computational complexity of the first-order estimator equations and the second-order estimator equation is the same, indicating no difference in online computations between the two estimators.

However, the second-order estimator is more elegant in terms of its mathematical structure as a second-order linear dynamical system and in terms of the prediction and filtered estimates' decoupling.

It is essential to note that this model is not a panacea, but rather a refinement of a previous model that may require further refinement in the future. This model is not the most general one that can be used for analysis purposes.

Overall, the high-order Kalman filter is a useful tool for state estimation in nonlinear systems. Mathematical derivations for the second order Kalman filter have been developed using two methods. The first method used estimation and prediction based on the innovations approach. While the second method was purely based on the second order discrete state variables model in matrix format.

In addition, as an application, we explored the development of second order Kalman filter-based on displacement/velocity estimators for undamped elastic systems. By employing a modal decomposition technique and exact discretization scheme, we obtained second-order vector recursions expressed solely in terms of displacements and velocities.

REFERENCES

[1] A. I. Iskanderani, “A New Approach to Linear Filtering and Prediction based on High order Models” Journal Technological Science & Engineering (JTSE), U.S. ISSN 2693 -1389 Vol. 2, No. 1, 2021.

[2] D. T. C. Allerton and B. M. Yu, "Innovations approach to second-order VDE signal models," IEEE Trans. on Signal Processing, vol. 57, no. 7, pp. 2552-2564, July 2009.

[3] B. M. Yu and D. T. C. Allerton, "Recursive second-order VDE signal models with innovations feedback," IEEE Trans. on Automatic Control, vol. 54, no. 9, pp. 2059-2074, Sept. 2009.

[4] B. M. Yu and D. T. C. Allerton, "Second-order VDE-based signal models with innovations feedback," in Proc. of the 47th IEEE Conf. on Decision and Control, Cancun, Mexico, Dec. 2008.

[5] J. R. Partington, "The innovations approach to second-order linear systems," SIAM J. Control and Optimization, vol. 44, no. 6, pp. 1963-1978, 2006.

[6] M. S. Grewal and A. P. Andrews, Kalman Filtering: Theory and Practice Using MATLAB, 3rd ed. Wiley-Interscience, 2008.

[7] L. Ljung, System Identification: Theory for the User, 2nd ed. Prentice-Hall, 1999.

[8] S. M. Kay, Fundamentals of Statistical Signal Processing: Estimation Theory, vol. 1. Prentice-Hall, 1993.

[9] Simon, D. (2006). Optimal state estimation: Kalman, H infinity, and nonlinear approaches. John Wiley & Sons.

[10] A. I. Iskanderani, and N.H. McClamroch, ’Signal Estimation for Second- Order Vector Difference Equation’s, IEEE Trans. on Autom. Control AC-30, 771-773, (1985).

[11] A. I. Iskanderani, Signal Estimation for Second-Order Vector Difference Equations (Filtering, Prediction, Higher-Order Systems), PHD Thesis, University of Michigan, 1984.

[12] Gelb, A. (1974). Applied Optimal Estimation. MIT Press.

Conflict of interest

“The authors declare that they have no conflict of interest.”

Iskanderani, Ahmed I. "Mathematical Derivation of the Second order Kalman filter Based on Estimation and Prediction." *The International Journal of Engineering and Science (IJES)*, 12(6), (2023): pp. 48-53.