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On ε-COAPPROXIMATION

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ABSTRACT

For a subset G of a metric space (X,d) and $\varepsilon>0$, an element $g_0 \in G$ is called an ε-coapproximation to $x \in X$ if $d(g_0,g) \le d(x,g)+\varepsilon$ for all $g \in G$. The set of all $\varepsilon-$coapproximations to x in G is denoted by $R_{G,\varepsilon}(x)$. In this paper , we discuss some basic properties and structure of the set of elements of $\varepsilon-$coapproximation.The underlying spaces are metric spaces or convex metric spaces or metric linear spaces . Directions for future research have also been discussed .

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The concept of elements of $\varepsilon-$approximation was introduced in normed linear spaces by Buck[2] under the name 'elements of good approximation'.Subsequently,many researchers discussed this concept (see [3],[4],[7] and references cited therein) . Thereafter , the concept of ε-coapproximation was introduced and discussed in normed linear spaces by Vaezpour et al. [9] . Some results on $\varepsilon-$coapproximation have also been discussed by As'ad and Ghazal [1] in normed linear spaces.This concept was extended to metric spaces in [3] and was discussed in metric linear spaces in [5] and [6].In this paper , we carry forward this study in spaces which are either metric spaces or metric linear spaces or convex metric spaces .

To start with , we recall a few definitions to be used in the sequel.

For a metric space (X,d) and a closed interval I=[0,1], a continous mapping $W: X\times X\times I \to X$ is said to be a convex structure on X if for all $x,y \in X$ and $\lambda \in I$,

$$d(u,W(x,y,\lambda)) \le \lambda d(u,x)+(1-\lambda)d(u,y)$$

for all $u \in X$. The metric space (X,d) together with a convex structure W , denoted by (X,d,W) is called a convex metric space [8].

A non – empty subset G of a convex metric space (X,d,W) is said to be

(i) convex if $W(x,y,\lambda) \in G$ for all $x,y \in G$ and $\lambda \in I$.

(ii) starshaped with star centre p , if there is some $p \in G$ such that $W(x,p,\lambda) \in G$ for all $x \in G$ and $\lambda \in I$.

A metric space (X,d) is called a metric linear space if (i) X is a linear space (ii) addition and scalar multiplications are continous in X , and (iii) d is translation invariant *i.e.* $d(x+z,y+z)=d(x,y)$ for all $x,y,z \in X$.

Let G be a subset of a metric space (X,d) and $\varepsilon>0$.An element $g_0 \in G$ is called an $\varepsilon-$approximation [2] ($\varepsilon-$coapproximation [3]) to $x \in X$ if

$$d(x,g_0) \le d(x,g)+\varepsilon \ (d(g_0,g) \le d(x,g)+\varepsilon) \text{ for all } g \in G.$$

For $x \in X$, the set of all $\varepsilon-$approximations($\varepsilon-$coapproximations)to x in G is denoted by $P_{G,\varepsilon}(x)(R_{G,\varepsilon}(x))$

$$i.e.\ P_{G,\varepsilon}(x)=\{g_0 \in G : d(x,g_0) \le d(x,g)+\varepsilon \text{ for all } g \in G\}$$

$R_{G,\varepsilon}(x) = \{g_0 \in G : d(g_0, g) \leq d(x,g) + \varepsilon \text{ for all } g \in G\}$.

For $\varepsilon > 0$, we obtain sets of best approximation (coapproximation)

The set G is called $\varepsilon -$proximinal($\varepsilon -$coproximinal) if $P_{G,\varepsilon}(x)(R_{G,\varepsilon}(x))$ is non – empty for each $x \in X$. It is said to be $\varepsilon -$Chebyshev ($\varepsilon -$coChebyshev) if $P_{G,\varepsilon}(x)(R_{G,\varepsilon}(x))$ contains exactly one element for every $x \in X$.

Since elements of $\varepsilon -$approximation always exist for every $\varepsilon > 0$, every non – empty subset of X is $\varepsilon -$proximinal. On the other hand , elements of $\varepsilon -$coapproximation may or may not exist . For $\varepsilon = 0$, $\varepsilon -$coproximinal and $\varepsilon -$coChebyshev sets are coproximinal and coChebyshev sets respectively. It is easy to see that $R_{G,\varepsilon}(x)$ is a closed set if G is a closed subset of X, $R_G(x) = \bigcap_{\varepsilon > 0} R_{G,\varepsilon}(x)$ and $R_{G,\varepsilon}(x) \subseteq R_{G,\delta}(x)$ for every $\delta \geq \varepsilon$.

For a linear subspace G of a metric linear space (X,d) and $\varepsilon > 0$, we define

$R^{-1}_{G,\varepsilon}(0) = \{x \in X : 0 \in R_{G,\varepsilon}(x)\}$.

For $x, y \in X$, we say that x is $\varepsilon -$orthogonal to y [9] , $x \perp_\varepsilon$ y if $d(x,0) \leq d(x, \alpha y) + \varepsilon$ for all real scalars α. For non – empty subsets A , B of X,we say that A is $\varepsilon -$orthogonal to B ; $A \perp_\varepsilon B$, if $a \perp_\varepsilon b$ for all $a \in$ A, $b \in$ B.

We define

$\check{G} = \{x \in X : G \perp x\}$

$\check{G}_\varepsilon = \{x \in X : G \perp_\varepsilon x\}$

$\equiv \{x \in X : g \perp_\varepsilon x \text{ for all } g \in G\}$

Clearly $\check{G} = \bigcap_{\varepsilon > 0} \check{G}_\varepsilon$.

Before proceeding further , we give few examples concerning elements of $\varepsilon -$coapproximation .

Example 1. Let $X = \mathbb{R}^2$ with Euclidean metric and $G = \{(x,y) : x^2 + y^2 = 1\}$.Then for $x = (0,0)$ and $\varepsilon = \frac{1}{2}$, we have

$R_{G,\frac{1}{2}}(0,0) = \{g_0 \in G : d(g_0, g) \leq d((0,0), g) + \frac{1}{2} \text{ for all } g \in G\}$

$$=\left\{g_0 \in G : d(g_0,g) \le \frac{3}{2} \text{ for all } g \in G\right\}$$

$$=\phi \ .$$

If we take $\varepsilon \ge 1$ then $R_{G,\varepsilon}(0,0)=G$.

Example 2 . For $\varepsilon>0$, let $X=\{(x,y)\in \mathbb{R}^2 : x^2+y^2=\varepsilon^2\}\cup\{(0,0)\}$ with Euclidean metric and

$G=\{(x,y): x^2+y^2=\varepsilon^2\}$. Then $R_{G,\varepsilon}(z)=G$ for each $z\in X$ *i.e.* G is $\varepsilon-$coproximinal . But G is not coproximinal as $R_G(0,0)=\phi$.

It is easy to see that $R_{G,\varepsilon}(x)$ may or may not be closed . However , it is closed if G is closed.

Example 3 [1] Let $X=\mathbb{R}^n$ with the norm

$$\|(x_1,x_2,\ldots,x_n)\| = |x_1|+|x_2|+\cdot\ \cdot\ \cdot+|x_n|$$

and $\varepsilon>0$, $G=\left\{(g_1,g_2,\ldots,g_n)\in \mathbb{R}^n :\ |g_i|<\frac{\varepsilon}{n} \text{ for all } 1\le i\le n\right\}$. Then G is ε-coproximinal but G is not closed .

We now discuss some basic properties of elements of $\varepsilon-$coapproximation , structure of sets of elements of co$-$approximation and directions for future research .

Proposition 1. If G is a convex subset of a convex metric space (X,d,W), $x\in X$ and $\varepsilon>0$, then $R_{G,\varepsilon}(x)$ is a convex subset of X .

Proof Let $g_1,g_2\in R_{G,\varepsilon}(x)$ and $0\le\alpha\le 1$. Consider

$$d\left[W(g_1,g_2,\alpha),\mathrm{g}\right] \le \alpha d(g_1,g)+(1-\alpha)d(g_2,g) \text{ for all } g\in G$$

$$\le \alpha\left[d(x,g)+\varepsilon\right]+(1-\alpha)\left[d(x,g)+\varepsilon\right] \text{ for all } g\in G$$

$= d(x,g)+\varepsilon$ for all $g \in G$.
Therefore, $d\left[W(g_1,g_2,\alpha),g\right] \le d(x,g)+\varepsilon$ for all $g \in G$ and $0 \le \alpha \le 1$.
This gives $W(g_1,g_2,\alpha) \in R_{G,\varepsilon}(x)$ as $W(g_1,g_2,\alpha) \in G$.

Hence $R_{G,\varepsilon}(x)$ is a convex set .

Proposition 2 Let G be a linear subspace of a metric linear space (X,d) and $\varepsilon > 0$. Then

(*i*) $g_0 \in R_{G,\varepsilon}(x)$ if and only if $0 \in R_{G,\varepsilon}(x-g_0)$,

(*ii*) $g_0 \in R_{G,\varepsilon}(x)$ if $x-g_0 \in \check{G}_\varepsilon$,

(*iii*) for $g_0 \in G$ and for any scalar α , we have $\alpha g_0 \in R_{G,\varepsilon}(\alpha x)$ if and only if $x-g_0 \in \check{G}_\varepsilon$,

(*iv*) $R_{G,\varepsilon}(x) = G \cap \left[x - R^{-1}_{G,\varepsilon}(0)\right]$,

(*v*) $R_{G,\varepsilon}(x) \supseteq \bigcup_{g_0 \in G}\left[\bigcap_{g \in G} P_{[g_0,x],\varepsilon}(g)\right]$, where $[g_0,x] \equiv \{\alpha x+(1-\alpha)g_0,\ |\alpha| \le 1\} \cap G$,

(*vi*) For $x \in X \mid G$, $\left\{g_0 \in G : \bigcap_{g \in G} P_{\langle g_0,x\rangle,\varepsilon}(g)\right\} \subseteq R_{G,\varepsilon}(x)$ where $\langle g_0,x\rangle = \{\alpha x+(1-\alpha)g_0, \alpha \in \mathbb{R}\}$.

Proof For proofs of (*i*) , (*ii*) and (*iii*) we refer to [6] .

(*iv*) $g_0 \in G \cap \left[x-R^{-1}_{G,\varepsilon}(0)\right] \Leftrightarrow g_0 \in G$ and $g_0 \in \left[x-R^{-1}_{G,\varepsilon}(0)\right]$

$\Leftrightarrow g_0 \in G$ and $g_0 = x-g_1, g_1 \in R^{-1}_{G,\varepsilon}(0)$

$\Leftrightarrow g_0 \in G$ and $g_1 = x-g_0 \in R^{-1}_{G,\varepsilon}(0)$

$\Leftrightarrow g_0 \in G$ and $x-g_0 \in R^{-1}_{G,\varepsilon}(0)$

$\Leftrightarrow g_0 \in G$ and $g_0 \in R_{G,\varepsilon}(x)$.

Therefore , $G \cap \left[x-R^{-1}_{G,\varepsilon}(0)\right] = R_{G,\varepsilon}(x)$.

(*v*) Let $h \in \bigcup_{g_0 \in G}\left[\bigcap_{g \in G} P_{[g_0,x],\varepsilon}(g)\right]$. Then $h \in P_{[g_0,x],\varepsilon}(g)$ for all $g \in G$ and for some $g_0 \in G$. This implies

$d(g,h) \le d(g,\alpha x+(1-\alpha)g_0)$ for all for all $g \in G$ and $|\alpha| \le 1$.

Therefore , $d(g,h) \le d(g,x)+\varepsilon$ for all $g \in G$, taking α=1 *i.e.* $h \in R_{G,\varepsilon}(x)$. Hence the result follows .

(*vi*) Let $g_0 \in \bigcap_{g \in G} P_{\langle g_0,x\rangle,\varepsilon}(g)$. Then $g_0 \in P_{\langle g_0,x\rangle,\varepsilon}(g)$ for all $g \in G$.

This implies

$$d(g_0, g) \le d(g, y) + \varepsilon \text{ for all } y \in \langle g_0, x \rangle \text{ and } g \in G$$

i.e. $d(g_0, g) \le d(g, \alpha x + (1-\alpha) g_0) + \varepsilon$ for all $g \in G$ and $\alpha \in$.

Therefore , $d(g_0, g) \le d(g, x) + \varepsilon$ for all $g \in G$, by taking α=1and so, $g_0 \in R_{G,\varepsilon}(x)$. The result follows .

Proposition 3. If G is a subset of a convex metric space (X, d, W) such that G is starshaped with respect to g_0 , then $R_{G,\varepsilon}(x)$ is starshaped with respect to g_0 provided $g_0 \in R_{G,\varepsilon}(x)$.

Proof Let $y \in R_{G,\varepsilon}(x)$. Then $d(y, g) \le d(x, g) + \varepsilon$ for all $g \in G$. Since G is starshaped with respect to g_0, $W(y, g_0, \lambda) \in G$ for all $\lambda \in$ I . We claim that $W(y, g_0, \lambda) \in R_{G,\varepsilon}(x)$. Consider

$$d[(W(y, g_0, \lambda), g] \le \lambda d(y, g) + (1-\lambda) d(g_0, g) \text{ for all } g \in G$$

$$\le \lambda[d(x, g) + \varepsilon] + (1-\lambda)[d(x, g) + \varepsilon] \text{ for all } g \in G$$

$$= d(x, g) + \varepsilon \text{ for all } g \in G .$$

This implies $W(y, g_0, \lambda) \in R_{G,\varepsilon}(x)$ for all $\lambda \in$ I and for all $y \in R_{G,\varepsilon}(x)$.Hence $R_{G,\varepsilon}(x)$ is starshaped with respect to g_0.

Proposition 4. If G is a subset of a convex metric space (X, d, W) , $g_0 \in R_{G,\varepsilon}(x)$ and $W(g_0, x, \lambda) \in G$ for some $\lambda \in$ I , then $W(g_0, x, \lambda) \in R_{G,\varepsilon}(x)$.

Proof Since $g_0 \in R_{G,\varepsilon}(x)$, $d(g_0, g) \le d(x, g) + \varepsilon$ for all $g \in G$.

Consider

$$d[(W(g_0, x, \lambda), g] \le \lambda d(g_0, g) + (1-\lambda) d(x, g) \text{ for all } g \in G$$

$$\le \lambda[d(x, g) + \varepsilon] + (1-\lambda) d(x, g) \text{ for all } g \in G$$

$$\le \lambda[d(x, g) + \varepsilon] + (1-\lambda)[d(x, g) + \varepsilon] \text{ for all } g \in G$$

$$= d(x, g) + \varepsilon \text{ for all } g \in G .$$

Since $W(g_0, x, \lambda) \in G$, we get $W(g_0, x, \lambda) \in R_{G,\varepsilon}(x)$.

Remarks

1. If G is $\varepsilon-$coChebyshev subset of a convex metric space (X,d,W), then $R_{G,\varepsilon}[W(x,R_{G,\varepsilon}(x),\lambda)]=R_{G,\varepsilon}(x)$ provided $W(x,R_{G,\varepsilon}(x),\lambda)\in G$.

2. We have the following reformulation of Proposition 4 :
If G is a subset of a convex metric space (X,d,W) and $x\in R^{-1}_{G,\varepsilon}(g_0)$, then $x\in R^{-1}_{G,\varepsilon}[W(g_0,x,\alpha)]$ if $W(g_0,x,\alpha)\in G$ and $\alpha\in[0,1]$.

Proposition 5. If G is a subset of a metric space (X,d) , then $R^{-1}_{G,\varepsilon}(g_0)\equiv\{x\in X:g_0\in R_{G,\varepsilon}(x)\}$ is a closed subset of X.
Proof Let x be a limit point of $R^{-1}_{G,\varepsilon}(g_0)$. Then there exists a sequence $\langle x_n\rangle$ in $R^{-1}_{G,\varepsilon}(g_0)$ such that $x_n\to x$. Since $g_0\in R_{G,\varepsilon}(x_n)$ for all n ,

$d(g_0,g)\le d(x_n,g)+\varepsilon$ for all $g\in G$.

This gives

$d(g_0,g)\le \lim d(x_n,g)+\varepsilon$ for all $g\in G$

i.e. $d(g_0,g)\le d(x,g)+\varepsilon$ for all $g\in G$

and so , $g_0\in R_{G,\varepsilon}(x)$ *i.e.* $x\in R^{-1}_{G,\varepsilon}(g_0)$.

Proposition 6. If G is a subset of a metric space (X,d) , then $R^{-1}_{G,\varepsilon}(g_0)=\bigcap_{g\in G} R^{-1}_{\{g_0,g\},\varepsilon}(g_0)$.

Proof Let $x \in R^{-1}_{G,\varepsilon}(g_0)$ and $g\in G$. Then $g_0 \in R_{G,\varepsilon}(x)$

and so

$d(g_0,g)\le d(x,g)+\varepsilon$ for all $g\in G$.

This inequality is obviously true for g_0.It follows that $x\in R^{-1}_{\{g_0,g\},\varepsilon}(g_0)$ for all $g\in G$. Therefore, $R^{-1}_{G,\varepsilon}(g_0)\subseteq \bigcap_{g\in G} R^{-1}_{\{g_0,g\},\varepsilon}(g_0)$.

Conversely, let $x\in \bigcap_{g\in G} R^{-1}_{\{g_0,g\},\varepsilon}(g_0)$ *i.e.* $x\in R^{-1}_{\{g_0,g\},\varepsilon}(g_0)$ for all $g\in G$. This gives

$d(g_0,g)\le d(x,g)+\varepsilon$ for all $g\in G$ and so $x\in R^{-1}_{G,\varepsilon}(g_0)$. Therefore , $\bigcap_{g\in G} R^{-1}_{\{g_0,g\},\varepsilon}(g_0)\subseteq R^{-1}_{G,\varepsilon}(g_0)$ and the proof is complete.

FUTURE DIRECTIONS

Analogous to the notions of $\varepsilon-$approximation and $\varepsilon-$coapproximation in the theory of best approximation, one can think of defining the concepts of $\varepsilon-$farthest and $\varepsilon-$cofarthest points in the theory of farthest points as under :

For a bounded subset K of a metric space (X,d) and $\varepsilon>0$, an element $k_0\in K$ is called an $\varepsilon-$farthest point($\varepsilon-$cofarthest point) to $x\in X$ if $d(x,k_0)\ge \delta(x,K)-\varepsilon$ $(d(k_0,k)\ge \delta(x,K)-\varepsilon)$ for all $k\in K$ where $\delta(x,K)\equiv \sup\{d(x,y):y\in K\}$. It is easy to see that the notion of $\varepsilon-$cofarthest points and so of co$-$farthest points (taking $\varepsilon=0$) are not meaningful.

It will be interesting to study $\varepsilon-$farthest points and prove results for $\varepsilon-$farthest points , similar to those for $\varepsilon-$approximation available in [2]$-$[4], [6] and [7].

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