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The First Integral Technique for Constructing the Exact Solution of Nonlinear Evolution Equations Arising in Mathematical Physics

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--***ABSTRACT***--
The exact solution of three nonlinear partial differential equations viz: Burgers-Fisher, Burgers-Huxley and modified Korteweg-de Vries equations are investigated theoretically using the first integral method. The equations were first converted to ordinary differential equation using the FIM routine and the resulting ODE was further transformed into a system of equations using new independent variable. With the aid of the Hilbert-Nullstellensatz theorem, the resulting system is solved for first integral solution to the ordinary differential equation.
KEYWORD: *Burgers-Fisher,* *Burgers-Huxley, Modified Korteweg-de Vries, Division theorem, Hilbert-Nullstellensatz*

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I. INTRODUCTION

The study of nonlinear sciences which is the branch of science that studies nonlinear partial differential equations (NLPDEs) which models physical phenomenon in science, engineering, hydrodynamics, Biology, chemistry, physics, optical fibre, plasma physics and chemical kinetics have enjoyed an intense period of attention over two decades now, Tascan et al [1], Feng [2]. Most of these NPDEs hitherto does not have close form or analytical solution, but due to the advent of powerful Computer technology, these PDEs can now be routinely solved for both approximate and exact solutions Wazwaz [3].

Due to the important nature of these NPDEs to both industry and the academia, several reliable and effective ansatz methods have been proposed to find the explicit and exact solutions. Some of these methods includes, trigonometric function series method Zhang [4], Ma and Fuchssteiner [5], the modified mapping and extended mapping method Liu et al [6], bifurcation method and the dynamical system approach Chen et al [7], Chen and Ma [8], the exp-function method Pinar and Yildirum [9], Sakthivel and Lee [10], the transformed rational method Lee and Ma [11], the Lie point symmetries consisting of the symmetry algebra Cheng and Ma [12], the linear superposition principle Fan and Ma [13], the tanh-coth and Banach contraction method Ebiwareme [14], Lie classical approach and $(G/G)-$ expansion Kumar et al [15], Lie transform perturbation method Nanayakkara [16], the trial expansion method Sonmezoglu et al [17], the modified tanh-coth method Prasad et al [18], the auxiliary equation method Zhang and Zhang [19], the homogenous balance method Abdulwahhab et al [20], the tanh method Parkes and Duffy [21], Malfiet et al [22], Raslan and Evans [23], the generalized hyperbolic function Zahran et al [24], Titan and Gao [25], Tan and Guo [26], the variable separation method Tang and Lou [27], Lou and Tang [28], the hyperbolic function method Xia and Wang [29], the exponential rational function Kaplan et al [30], sub-equation method Zhang et al [31], Shasha et al [32].

The first integral method was first proposed by Feng [33] by applying the ring theory of commutative algebra to present elegant treatment of exact solutions of NPDEs. In the first integral method, the division theorem is used to find the first integral solution in explicit form that has polynomial coefficients. FIM has caught the attention of many researchers because it

gives explicit exact solutions of NPDEs without complex and lengthy calculations Javeed et al [34], Waheed et al [35]. The first integral method has been extensively applied to solve diverse nonlinear partial differential equations Feng [36], Feng [37], Wang and Feng [38], Li and Guo [39], Mirzazadeh et al [40], Jafar et al [41], Elsayed and Yasser [42], Aslan [43].

In this article, we employ the first Integral method based on the theory of commutative algebra to seek for exact solutions of the Burgers-Fisher, Burgers-Huxley and modified Korteweg-de Vries equations. From both previous and contemporary literatures, these equations have not been studied before. The result reveal, the method is reliable, efficient, easily computable and widely applicable to solving a large class of nonlinear partial differential equations in engineering and sciences. The benchmark solution is in agreement with those in literature and provide a reference guide to other problems.

II. FENG INTEGRAL METHOD (FIM)

The first integral method based on the ring theory of commutative algebra, proposed by Feng is a direct algebraic method for obtaining exact solutions of nonlinear partial differential equations. This method is applicable to both integrable and nonintegrable equations.

Following Aslan [43], the basics of the first Integral method are summarised as follows

Step 1. Consider a nonlinear PDE of the form

$$P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, \dots\right) = 0 \tag{1}$$

Step 2. Using the wave transformation of the form

$$u(x,t) = f(z), \;\; z = x - ct \tag{2}$$

Where $u(x,t)$ is the solution of the PDE in Eq. (1)

Step 3. In view of Step 2, the change in derivative with respect to the independent variables, x and t

$$\left.\begin{aligned} \frac{\partial}{\partial t}(.) &= -c\frac{d}{dz}(.) \\ \frac{\partial}{\partial x}(.) &= \frac{d}{dz}(.) \\ \frac{\partial^2}{\partial x^2}(.) &= \frac{d^2}{dz^2}(.) \end{aligned}\right\} \tag{3}$$

Step 4. Using step 3, the PDE in Eq. (1) changes to an ODE in the form

$$Q\left(f, \frac{df}{dz}, \frac{d^2 f}{dz^2}, \dots\dots\right) = 0 \tag{4}$$

Step 5. Introducing a new independent variable

$$X(z) = f(z), \;\; Y = \frac{\partial f(z)}{\partial z} \tag{5}$$

Step 6. Utilizing step 5 leads to a system of an ODE of the form

$$\left.\begin{aligned} \frac{dX(z)}{dz} &= Y(z) \\ \frac{dY(z)}{dz} &= F\big(X(z), Y(z)\big) \end{aligned}\right\} \tag{6}$$

Division Theorem

Suppose that $Q(x,y)$ and $R(x,y)$ be polynomials in the complex domain $C[x,y]$, and $Q(x,y)$ is irreducible in $C[x,y]$. If $R(x,y)$ vanishes at all zero points of $Q(x,y)$, then there exists a polynomial $H(x,y)$ in $C[x,y]$ such that $R(x,y) = Q(x,y)H(x,y)$

Hilbert-Nullstellensatz Theorem

Let k be a field and L be an algebraic closure of k

(i) Every ideal γ of $k[X_1, \dots., X_n]$ not containing 1 admits at least one zero in $\boldsymbol{L}^n$

(ii) Let $x = (x_1, \dots.., x_n)$ and $y = (y_1, \dots.., y_n)$ be two elements of $\boldsymbol{L}^n$; for the set of polynomials of $K[X_1, \dots.., X_n]$ zero at x to be identical with the set of polynomial s of $k[X_1, \dots\dots, X_n]$ zero at y, it is necessary and sufficient that there exists a k −automorphism s of L such that $y_i = s(x_i)$ for $1 \le i \le n$.

(iii) For an ideal α of $k[X_1, \dots.., X_n]$ to be maximal, it is necessary and sufficient that there exists an x in $\boldsymbol{L}^n$ such that α is the set of polynomials of $k[X_1, \dots\dots., X_n]$ zero of x.

(iv) For a polynomial Q of $k[X_1, \dots\dots., X_n]$ to be zero on the set of zeros in $\boldsymbol{L}^n$ of an ideal γ of $k[X_1, \dots.., X_n]$, it is necessary and sufficient that there exists an integer $m > 0$ such that $Q^m \in \gamma$.

In the ring theory of commutative algebra, the division theorem follows from the Hilbert-Nullstellensatz theorem. The central idea of the FIM method is to construct a first integral with polynomial coefficients of an explicit form an equivalent autonomous planar system, using the division theorem in the complex domain to Eq. (1), which can reduce Eq. (4) to a first-order
integrable ordinary differential equation. An exact solution to Eq. (6) can then be obtained directly by solving the equation

III. NUMERICAL APPLICATIONS

In this section, the exact solutions of three nonlinear partial differential equations namely: Burgers-Fisher, Burgers-Huxley and modified Korteweg-de Vries equations are presented using the first integral method. The result shows the method is flexible, accurate, efficient and has wide applicability, which converges rapidly to the exact solution.

Example 3.1 **The Burgers-Fisher Equation**

In the description of physical phenomena on nonlinear sciences especially in Mathematical physics and engineering, partial differential equations are pivotal to model the interaction between reaction mechanism, convective effects and diffusion transport Rabboh et al [44]. Burgers-Fisher equation which is the prototypical model has useful applications in the field of gas dynamics, number theory, heat conduction, financial mathematics, physics application, traffic flow and fluid mechanics Chandraker et al [45], Kocacoban et al [46].
Consider the Burgers-Fisher equation as follows

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} - u(1-u) = 0$$

Rearranging the above we have

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x} + u(1-u) \tag{7}$$

Using the wave transformation

$$u(x,t) = f(z), z = x - ct \tag{8}$$

Putting Eq. (8) into Eq. (7), we obtain

$$-c\frac{df(z)}{dz} = \frac{d^2 f(z)}{dz^2} - f(z)\frac{df(z)}{dz} + f(z)\big(1 - f(z)\big) \tag{9}$$

Using Eq. (5), the above equation reduced to the form

$$-cY(z) = Y'(z) - X(z)Y(z) + X(z)\big(1 - X(z)\big) \tag{10}$$

With resulting first order differential equations below

$$\frac{dX(z)}{dz} = Y(z) \tag{11}$$

$$\frac{dY(z)}{dz} = -cY(z) + X(z)Y(z) - X(z)\big(1 - X(z)\big) \tag{12}$$

According to the first integral method, let us assume that $X(z)$ and $Y(z)$ be nontrivial solutions of Eqs (11) and (12) and $q(X,Y) = \sum_{k=0}^{m} a_k(X)Y^k = 0$ be an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q[X(z),Y(z)] = \sum_{k=0}^{m} a_k(X)Y^k = 0 \tag{13}$$

Where $a_k(X),\ 0 \le k \le m$ is a polynomial of X and $a_k(X) \neq 0$
Eq. (13) is the first integral of Eqs. (11) and (12)
Assuming that $k = 2$ in Eq. (13). Owing to the Division Theorem, there exists a polynomial of the form $g(X) + h(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dq}{dz} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial z} = (g(X) + h(X)Y)\sum_{k=0}^{2} a_k(X)Y^k \tag{14}$$

For the analysis of this problem, we consider only two possible cases of $k = 1$ and 2 in Eq. (13)

Case 1. Putting $k = 1$ in Eq. (14) and comparing the coefficients of Y^k $(k = 0,1,2)$ on both sides of Eq. (14), we obtain the following equations.

$$Y^0:\ g(X)a_0(X) = a_1(X)\{X(z)[(X - c)X + X - 1]\} \tag{15}$$

$$Y^1:\ \frac{da_0(X)}{dX} = g(X)a_1(X) + \mathrm{h}(X)a_0(X) \tag{16}$$

$$Y^2:\ \frac{da_1(X)}{dX} = h(X)a_1(X) \tag{17}$$

Since $a_k(X)$ $(K = 0,1,2)$ are polynomials, then we deduce from Eq. (14) that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that deg $\big(g(X)\big) = 1$ only.
Now suppose that
$g(X) = A_1X + B_0$ and $A_1 \neq 0$, we find that $A_0(X)$ on integration gives

$$a_0(X) = \frac{A_1}{2}x^2 + B_0X + A_0 \tag{18}$$

Putting the values of $a_0(X), a_1(X)$ and $g(X)$ into Eq. (15) and setting all coefficients of the powers of X to zero, we obtain a nonlinear system of algebraic equation in the form

$$X:\ A_0A_1 + B_0^2 = -1 \tag{19}$$

$$X^2:\ \frac{3}{2}A_1B_0 = 1 - c \tag{20}$$

$$X^3:\ \frac{A_1^2}{2} = 1 \tag{21}$$

Solving the systems in Eqs. $(19) - (21)$ gives the constants

$$A_0 = \frac{-9-2(1-c)^2}{9\sqrt{2}},\quad A_1 = \sqrt{2},\quad B_0 = \frac{\sqrt{2}(1-c)}{3} \tag{22}$$

Putting Eq. (22) into Eq. (13), we obtain the solution for $Y(z)$ in the form

$$Y(z) = \frac{-9-2(1-c)^2}{9\sqrt{2}} + \sqrt{2}\,X^2(z) \tag{23}$$

Combining Eq. (23) with (11), the exact solution of Eq. (7) become

$$X(z) = \frac{3}{\sqrt{2}} X^3(z) \tag{24}$$

Hence the solitary solution to the Burgers-Huxley equation is given by

$$u(x,t) = \frac{3}{\sqrt{2}} X^3(x - ct) \tag{25}$$

Case II: Putting $k = 2$ and equating the coefficients of $Y^k s$ $(k = 0,1,2,3)$ on both sides of Eq. (14), we get the following

$$a_1(X)\dot{Y} = g(X)a_0(X) = a_1(X)\{X(z)[(X - c)X + X - 1]\} \tag{26}$$

$$\frac{da_0(X)}{dX} = -2a_2(X)\{X(z)[(X - c)X + X - 1]\} + g(X)a_1(X) + h(X)a_0(X) \tag{27}$$

$$\frac{da_1(X)}{dX} = g(X)a_2(X) + h(X)a_1(X) \tag{28}$$

$$\frac{da_2(X)}{dX} = h(X)a_2(X) \tag{29}$$

Assuming $a_2(X)$ is a polynomial of X, then from Eq. (29), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. To simplify the calculation, we take $a_2(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, hence we conclude that $deg\big(g(X)\big) = 1$ only. Thus, we assume that $g(X) = A_1(X) + B_0$ and $A_1 \neq 0$, then the values of the remaining constant, $a_1(X)$ and $a_0(X)$ as

$$a_1(X) = \frac{A_1}{2} X^2 + B_0 X + A_0 \tag{30}$$

$$a_0(X) = \left(\frac{A_1^2}{8}\right) X^4 + \left(\frac{A_1 B_0}{2} - \frac{2}{3}\right) X^3 + \left(\frac{A_1 A_0}{2} + \frac{B_0^2}{2} + c - 1\right) X^2 + (A_0 B_0 + 2)X + \alpha \tag{31}$$

Substituting the values of $a_0(X), a_1(X), a_2(X)$ and $g(X)$ into Eq. (26) and setting all the coefficients of powers of X to zero, then a system of nonlinear algebraic equation is obtained. Solving these equations yield the constants,

$$B_0 = 0,\ A_0 = \frac{11}{3}, A_1 = 1, c = \frac{7}{3} \text{ and } \alpha = -\frac{11}{3} \tag{32}$$

Putting Eq. (32) into Eq. (13), we obtain

$$Y(z) = \frac{11}{3} + X^2(z) \tag{33}$$

Combining Eqs. (33) and (11), the resulting exact solution for the equation become

$$u(x,t) = \frac{11}{3} X + \frac{X^3}{3}\left(x - \frac{7}{3} t\right) \tag{34}$$

Example 3.2 **The Burgers-Huxley Equation**

This is a nonlinear advection-diffusion partial differential equation which model reaction mechanism, transport and nerve propagation of waves in diverse fields of science like Physics, economics and ecology Murray [47], Zhu et al [48]. It is applicable in acoustic turbulence, hydrodynamic theory, traffic flow, general mechanics, chemistry, metallurgy, mathematics, engineering. Satsuma [49], Timilehin and Adedapo [50]. The basics of the Burgers-Huxley equation is given below

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u\frac{\partial u}{\partial x} + u(k - u)(u - 1), k \neq 0 \tag{35}$$

Taking $k = 1$, the above equation in rearranged form become

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} - u(u - 1)^2 \tag{36}$$

Let $u(x,t) = f(z), z = x - ct$ be a solution of Eq. (36) (37)

$$-c\frac{df(z)}{dz} = \frac{d^2 f(z)}{dz^2} + f(z)\frac{df(z)}{dz} - f(z)(f(z) - 1)^2 \tag{38}$$

Introducing a new independent variable of the form

$$X(z) = Y(z), Y(z) = \frac{\partial f(z)}{\partial z} \tag{39}$$

The equation now reduced to the system of equations of the form

$$\frac{\partial X(z)}{\partial z} = Y(z) \tag{40}$$

$$\frac{\partial Y(z)}{\partial z} = \big(-c - X(z)\big)Y(z) + X(z)(X(z) - 1)^2 \tag{41}$$

According to the first integral method, let us assume that $X(z)$ and $Y(z)$ be nontrivial solutions of Eqs (40) and (41) and $q(X,Y) = \sum_{k=0}^{m} a_k(X)Y^k = 0$ be an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q[X(z), Y(z)] = \sum_{k=0}^{m} a_k(X)Y^k = 0 \tag{42}$$

Where $a_k(X),\ 0 \leq k \leq m$ is a polynomial of X and $a_k(X) \neq 0$

Eq. (42) is called the first integral of Eqs. (40) and (41)

Assuming that $k = 2$ in Eq. (42). According to the Division Theorem, there exists a polynomial of the form $g(X) + h(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dq}{dz} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial z} = (g(X) + h(X)Y)\sum_{k=0}^{2} a_k(X)Y^k \tag{43}$$

For the analysis that follow, we consider two possible cases of $k = 1$ and 2 in Eq. (42)

Case 1. Setting $k = 1$ in Eq. (43) and compare coefficients of Y^k $(k = 0,1,2)$ on both sides, we obtain the following equations.

$$\frac{da_1(X)}{dX} = h(X)a_1(X) \tag{44}$$

$$\frac{da_0(X)}{dX} = g(X)a_1(X) + \mathrm{h}(X)a_0(X) \tag{45}$$

$$a_1(X)[(-c - X)X + X(X-1)^2] = g(X)a_0(X) \tag{46}$$

Clearly $a_k(X)$ $(K = 0,1,2)$ are polynomials, so we deduce from Eq. (43) that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that deg $\left(g(X)\right) = 1$ only.

Now suppose that
$g(X) = A_1X + B_0$ and $A_1 \neq 0$, we find that $A_0(X)$ on integration yield

$$a_0(X) = A_0 + B_0X + \frac{A_1}{2}x^2 \tag{47}$$

Substituting the values of $a_0(X), a_1(X)$ and $g(X)$ into Eq. (44) and setting all coefficients of the powers of X to zero, we obtain a nonlinear system of algebraic equation in the form

$$X{:}\ \ A_0A_1 + B_0^2 = 1 - c \tag{48}$$

$$X^2{:}\ \ \frac{3}{2}A_1B_0 = -3 \tag{49}$$

$$X^3{:}\ \ \frac{A_1^2}{2} = 1 \tag{50}$$

Solving the system in Eqs. $(47) - (50)$ yield the constants

$$A_0 = 0,\ \ A_1 = \sqrt{2},\quad B_0 = -\sqrt{2},\ c = -1 \tag{51}$$

Putting Eq. (51) into Eq. (42), we obtain the solution for $Y(z)$ in the form

$$Y(z) = \sqrt{2}\left(1 - X^2(z)\right) \tag{52}$$

Combining Eq. (52) with (40), the exact solution of Eq. (36) become

$$X(z) = \frac{\sqrt{2}}{3}\left(3 - X^3(z)\right) \tag{53}$$

Hence the solitary solution to the Burgers-Huxley equation is given by

$$u(x,t) = \frac{\sqrt{2}}{3}\left(3 - X^3(x+t)\right) \tag{54}$$

Case II: Setting $k = 2$ and equating the coefficients of Y^is on both sides of Eq. (43), we get the following

$$a_1(X)\dot{Y} = g(X)a_0(X) = a_1(X)[(-c - X)X + X(X-1)^2] \tag{55}$$

$$\frac{da_0(X)}{dX} = -2a_2(X)\{X(z)[(X-c)X + X - 1]\} + g(X)a_1(X) + h(X)a_0(X) \tag{56}$$

$$\frac{da_1(X)}{dX} = g(X)a_2(X) + h(X)a_1(X) \tag{57}$$

$$\frac{da_2(X)}{dX} = h(X)a_2(X) \tag{58}$$

Assuming $a_2(X)$ is a polynomial of X, then from Eq. (58), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. To simplify the calculation, we take $a_2(X) = 1$. Balancing the degree of $g(X)$ and $a_0(X)$, we conclude that $deg\left(g(X)\right) = 1$ only. Thus, we assume that $g(X) = A_1(X) + B_0$ and $A_1 \neq 0$, then the values of the remaining constant become.

$$a_1(X) = A_0 + B_0X + \frac{A_1}{2}X^2 \tag{59}$$

$$a_0(X) = \left(\frac{A_1^2}{8} + \frac{1}{4}\right)X^4 + \left(\frac{A_1B_0}{2} - \frac{4}{3}\right)X^3 + \left(\frac{A_1A_0}{2} + \frac{B_0^2}{2} + c + \frac{1}{2}\right)X^2 + A_0B_0X + \beta \tag{60}$$

Substituting the values of $a_0(X), a_1(X), a_2(X)$ and $g(X)$ into Eq. (55) and setting all the coefficients of powers of X to zero, then a system of nonlinear algebraic equation is obtained. Solving these equations yield the constants,

$$B_0 = \frac{5\sqrt{2}}{3},\ A_0 = \frac{\sqrt{2}}{3}, A_1 = \sqrt{2}, c = -\frac{41}{9} \text{ and } \beta = 0 \tag{61}$$

Putting Eq. (61) into Eq. (42), we obtain

$$Y(z) = \frac{\sqrt{2}}{3} + \sqrt{2}X(z) + \frac{5\sqrt{2}}{3}X^2(z) \tag{62}$$

Combining Eqs. (62) and (40), the resulting exact solution for the equation become

$$u(x,t) = \frac{11}{3}X + \frac{X^3}{3}\left(x - \frac{7}{3}t\right) \tag{63}$$

3.3 The Modified Korteweg-de Vries Equation (mKDV)

The modified Korteweg-de Vries equation denoted (mKDV) is a variant of the traditional KDV equation which differs only in the nonlinear term but includes the dispersion term (u_{xxx}).

$$u_t + 6u^2u_x + u_{xxx} = 0 \tag{64}$$

We seek a travelling wave transformation of Eq. (64) as follows

$$u(x,t) = f(z),\ z = x - ct \tag{65}$$

Putting Eq. (65) into Eq. (64), the PDE above is transformed into an ODE of the form

$$-c\frac{df(z)}{dz} + 6\big(f(z)\big)^2\frac{df(z)}{dz} + \frac{d^3f(z)}{dz^3} = 0$$

$$-cf'(z) + 6\big(f(z)\big)^2 f'(z) + f'''(z) = 0 \tag{66}$$

Integrating both sides of Eq. (66), we obtain

$$-cf(z) + 2\big(f(z)\big)^3 + f''(z) = 0 \tag{67}$$

Rearranging Eq. (67), we get the highest derivative in the form

$$f''(z) = cf(z) - 2\big(f(z)\big)^3 \tag{68}$$

Introducing a new independent variable using Eq. (5) of the form

$$X(z) = Y(z), Y(z) = \frac{\partial f(z)}{\partial z} \tag{69}$$

The resulting system of equations takes the form

$$\frac{\partial X(z)}{\partial z} = Y(z) \tag{70}$$

$$\frac{\partial Y(z)}{\partial z} = cX(z) - 2\big(X(z)\big)^3 \tag{71}$$

According to the first integral method, let us assume that $X(z)$ and $Y(z)$ be nontrivial solutions of Eqs (70) and (71) and $q(X,Y) = \sum_{k=0}^{m} a_k(X)Y^k = 0$ be an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$Q[X(z),Y(z)] = \sum_{k=0}^{m} a_k(X)Y^k = 0 \tag{72}$$

Where $a_k(X),\ 0 \le k \le m$ is a polynomial of X and $a_k(X) \neq 0$

Eq. (72) is called the first integral of Eqs. (70) and (71)

Assuming that $k = 2$ in Eq. (72). According to the Division Theorem, there exists a polynomial of the form $g(X) + h(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dQ}{dz} = \frac{\partial Q}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial Q}{\partial Y}\frac{\partial Y}{\partial z} = \big(g(X) + h(X)Y\big)\sum_{k=0}^{2} a_k(X)Y^k \tag{73}$$

For the analysis that follow, we consider two possible cases of $k = 1$ and 2 in Eq. (72)

Case 1. Setting $k = 1$ in Eq. (73) and compare coefficients of Y^k $(k = 0,1,2)$ on both sides, we obtain the following equations.

$$\frac{da_1(X)}{dX} = h(X)a_1(X) \tag{74}$$

$$\frac{da_0(X)}{dX} = g(X)a_1(X) + \mathrm{h}(X)a_0(X) \tag{75}$$

$$a_1(X)\left[cX(z) - 2\big(X(z)\big)^3\right] = g(X)a_0(X) \tag{76}$$

Clearly $a_k(X)$ $(K = 0,1,2)$ are polynomials, so we deduce from Eq. (73) that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that deg $\big(g(X)\big) = 1$ only.

Now suppose that

$g(X) = A_1X + B_0$ and $A_1 \neq 0$, we find that $A_0(X)$ on integration yield

$$a_0(X) = A_0 + B_0X + \frac{A_1}{2}x^2 \tag{77}$$

Substituting the values of $a_0(X), a_1(X)$ and $g(X)$ into Eq. (74) and setting all coefficients of the powers of X to zero, we obtain a nonlinear system of algebraic equation in the form

$$X:\ A_0A_1 + B_0^2 = c \tag{78}$$

$$X^2:\ \frac{3}{2}A_1B_0 = 0 \tag{79}$$

$$X^3:\ \frac{A_1^2}{2} = -2 \tag{80}$$

Solving the system in Eqs. $(77) - (80)$ yield the constants

$$A_0 = \frac{c}{2i} \qquad A_1 = 2i \quad B_0 = 0,\ c = 0 \tag{81}$$

Putting Eq. (81) into Eq. (72), we obtain the solution for $Y(z)$ in the form

$$Y(z) = \frac{c}{2i} + 2iX(z) \tag{82}$$

Combining Eq. (82) with (70), the exact solution of Eq. (64) become

$$X(z) = \frac{1}{2i}\big(c + 4X(z)\big) \tag{83}$$

Hence the solitary solution to the Burgers-Huxley equation is given by

$$u(x,t) = \pm\sqrt{c}\sec\sqrt{c}\,(x - ct) \tag{84}$$

Case II: Setting $k = 2$ and equating the coefficients of Y^is on both sides of Eq. (73), we get the following

$$a_1(X)\dot{Y} = g(X)a_0(X) = a_1(X)\left[cX(z) - 2\big(X(z)\big)^3\right] \tag{85}$$

$$\frac{da_0(X)}{dX} = -2a_2(X)\left[cX(z) - 2\big(X(z)\big)^3\right] + g(X)a_1(X) + h(X)a_0(X) \tag{86}$$

$$\frac{da_1(X)}{dX} = g(X)a_2(X) + h(X)a_1(X) \quad (87)$$

$$\frac{da_2(X)}{dX} = h(X)a_2(X) \quad (88)$$

Assuming $a_2(X)$ is a polynomial of X, then from Eq. (88), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. To simplify the calculation, we take $a_2(X) = 1$. Balancing the degree of $g(X)$ and $a_0(X)$, we conclude that $deg(g(X)) = 1$ only. Thus, we assume that $g(X) = A_1(X) + B_0$ and $A_1 \neq 0$, then the values of the remaining constant become.

$$a_1(X) = A_0 + B_0X + \frac{A_1}{2}X^2 \quad (89)$$

$$a_0(X) = \left(\frac{A_1^2}{8} + 1\right)X^4 + \left(\frac{A_1B_0}{2}\right)X^3 + \left(\frac{A_1A_0}{2} + \frac{B_0^2}{2} - c\right)X^2 + A_0B_0X + d \quad (90)$$

Substituting the values of $a_0(X), a_1(X), a_2(X)$ and $g(X)$ into Eq. (85) and setting all the coefficients of powers of X to zero, then a system of nonlinear algebraic equation is obtained. Solving these equations yield the constants,

$$B_0 = 0,\ A_0 = \sqrt{2}, A_1 = 0, c = 2 \text{ and } d = 0 \quad (91)$$

Putting Eq. (91) into Eq. (72), we obtain

$$Y(z) = \sqrt{2} + \frac{\sqrt{3}}{2} X^2(z) \quad (92)$$

Combining Eqs. (92) and (70), the resulting exact solution for the equation become

$$u(x,t) = \sqrt{2} + \frac{\sqrt{3}}{2}(x - 2t) \quad (93)$$

IV. CONCLUDING REMARKS

In this paper, three nonlinear evolution partial differential equations have been investigated using the first integral method. The basics of the method was extensively discussed and its efficiency has been confirmed by applying it to the selected problems. The results obtained show that the method is valid, efficient, powerful, applicable and showed ability to handle several other nonlinear equations

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