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Different Solutions to a Mathematical Problem: A Case Study of Calculus 12

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ABSTRACT
An important duty of the mathematics teacher is to train and to develop thinking for students. To accomplish this duty, teachers can organize creative activities for students through activities of solving problems. In particular, there is an effective way to train students to think is that teachers can organize activities of solving problems in many different ways. Based on this idea, we implement an experiment for students in grade 12 to calculate integrals in various ways. The results of the study showed that students were active to find out different solutions to the given problem.
Keywords: *Calculus teaching, mathematics education, multi-solution to a problem, problem solving*

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I. INTRODUCTION

Solving a problem in different ways has an important significance for the teaching of mathematics. Polya (1973, 6.61) insisted that encouraging students to get different results can help them obtain even the more elegant solutions. To be able to solve the problem in different ways, students need to mobilize as much relevant knowledge, then they continue to analyze, synthesize and evaluate the facts of the problem and finally find a reasonable explanation. That process requires students to use the necessary thinking operations and logical reasoning. Silver, Ghousseini, Gosen, Charalambous & Strawhun, (2005, p.228) argued that different solutions can facilitate the connection of a problem with the various elements of the knowledge with which a student may be familiar, thus leading to strengthen the networks of related ideas.
In reality, the current calculus 12 (Giải tích 12) textbook in Vietnam provides students with many problems about calculating integrals (Hao, 2006). However, in our study, there is no any problem that requires students to solve in many different ways. In this case study, we want to find out the answer to the following question:

The research question: *For a mathematical problem with requirement of solving in different ways, are students active to find out various solutions to the problem?*

II. METHODOLOGY

2.1. Participants

The experiment was carried out in class 12A2 at the high school of Tran Ngoc Hoang, Can Tho City.
This class consisted of 31 students.
Time: started at 9 45 and ended 10:00, 01/29/2015.

2.2. Instrument and procedure

Students were requested to solve the following problem:

Calculate the following integral in different ways: $I = \int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{4} - x\right) dx$.

Students worked individually (15 minutes). Students had to solve the problem printed on a paper. Every student's behavior was shown on assignments. More specifically, students showed their ability in dealing with solving the problem in various ways.

2.2.3. Pre-analyzing the problem

a. The class context of the experimental problem
The problem was organized for students after they had completed the primitive and integral topics. In particular, the methods of finding primitive were mentioned.
b. The solutions expected to the problem

Table 1 shows the possible solutions to the problem

Table 1: The solutions expected to the problem

Code	Solution strategy	Solution
S1	Applying the definition of primitive	$I = \int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{4} - x\right) dx = \int_0^{\frac{\pi}{2}} \left(-\cos\left(\frac{\pi}{4} - x\right)\right)' dx = -\cos\left(\frac{\pi}{4} - x\right)\Big\|_0^{\frac{\pi}{2}}$ $= -\cos\left(-\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$
S2	Changing the function under the integral sign in the supply side	$I = \int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{4} - x\right) dx = \int_0^{\frac{\pi}{2}} \cos\left(\frac{\pi}{4} + x\right) dx = \sin\left(\frac{\pi}{4} + x\right)\Big\|_0^{\frac{\pi}{2}}$ $= \sin\frac{3\pi}{4} - \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0$
S3	Using the relationship between the odd property of function and its integral on an interval.	Let $t = \frac{\pi}{4} - x \Rightarrow dx = -dt$. Therefore, $x = 0 \Rightarrow t = \frac{\pi}{4}$, $x = \frac{\pi}{2} \Rightarrow t = -\frac{\pi}{4}$ $I = -\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \sin t dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin t dt$ $f(t) = \sin t$ is odd on $\left(-\frac{\pi}{4}; \frac{\pi}{4}\right)$, then $I = -\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \sin t dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin t dt = 0$
S4	Using derivative spellings	$I = \int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{4} - x\right) dx = -\int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{4} - x\right) d\left(\frac{\pi}{4} - x\right) = \cos\left(\frac{\pi}{4} - x\right)\Big\|_0^{\frac{\pi}{2}}$ $= \cos\left(-\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$
S5	Changing the function under the integral sign in the addition formula	$I = \int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{4} - x\right) dx = \int_0^{\frac{\pi}{2}} \left(\sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x\right) dx = \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{2}} (\cos x - \sin x) dx$ $= \frac{\sqrt{2}}{2}(\sin x + \cos x)\Big\|_0^{\frac{\pi}{2}} = \frac{\sqrt{2}}{2}\left(\sin\frac{\pi}{2} + \cos\frac{\pi}{2} - \sin 0 - \cos 0\right) = 0$
S6	Changing variables, changing the limits of the integral	Let $t = \frac{\pi}{4} - x \Rightarrow dx = -dt$. Therefore, $x = 0 \Rightarrow t = \frac{\pi}{4}$, $x = \frac{\pi}{2} \Rightarrow t = -\frac{\pi}{4}$ $I = -\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \sin t dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin t dt = -\cos t\Big\|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = -\cos\frac{\pi}{4} + \cos\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$
S7	Changing variables, but do not change the limits of the integral	Let $t = \frac{\pi}{2} - x \Rightarrow dt = -dx$. Therefore, $x = 0 \Rightarrow t = \frac{\pi}{2}$, $x = \frac{\pi}{2} \Rightarrow t = 0$ $I = -\int_{\frac{\pi}{2}}^{0} \sin\left(t + \frac{\pi}{4}\right) dt = \int_0^{\frac{\pi}{2}} \sin\left(t + \frac{\pi}{4}\right) dt$

		$= -\cos\left(t+\frac{\pi}{4}\right)\Big\vert_0^{\frac{\pi}{2}} = -\cos\frac{3\pi}{4} + \cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$

III. RESULTS AND DISCUSSION

Table 2: Statistics of the number of students with the right strategies, and the right solutions

	S1	S2	S3	S4	S5
Number of students	29 (93.5%)	14 (45.2%)	31 (100%)	18 (58.1%)	31 (100%)
Number of students doing correctly	9 (31%)	12 (85.7%)	14 (45.2%)	17 (94.4%)	28 (90.3%)

The Table 2 indicated that the dominant solutions belonged to S5 (31/31 students, at 100% respectively). In general, the participants were in the right direction, i.e., using integral's linear property by converting the given integral into addition (or subtraction) of the two integrals which might be used directly in the primitive table. Moreover, some of them also added the formula $\sin(a-b) = \sin a\cos b - \cos a\sin b$ in their work. This could be explained easily since these participants were familiar with applying the theoretical basis in solving problems. Three students (representing 9.68 %) had inaccurate answers.

The solution of S3 was also as good as that of S5 (31/31 students, approximately 100%). However, only 14 students (representing 45.16 %) had the correct answers when choosing this strategy. Most errors were caused by using wrong values after changing integral variables. Understanding from the concept of definite integrals on $[a;b],\ a<b$ might lead learners to this mistake though they had been noted that: "In the case of $a=b$ or $a>b$, there is a convention that $\int_a^a f(x)\,dx = 0;\ \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ " , [2 , p.105] .

S1 solution was also chosen by many participants (29/31 students, accounted for 93.55%). In general, learners had mastered primitive concept and consciously had a focus on this method. Though so, only 9 students (representing 31.03%) got the correct answer. This might be apparently explained as in the current textbook program, assignments can be solved through specifically formulas in the primitive table, i.e., changing variables or solving through another unknown factor. Students do not probably have many opportunities to practice this method when they do not practice finding different ways to get a problem result.

S4 solution was also taken by a number of learners (18/31 students, accounted for 58.06%). This was the highest rate for those who had the right answer (17/18 students, accounted for 94.44%) when selecting and presenting solutions by this method. Reflecting on the advantages (and disadvantages) of each solution in the survey showed the superiority of differential method.

S2 solution was the least one to be chosen, because most students claimed that there was no difference between two primitive functions: $\sin\left(\frac{\pi}{4}-x\right)$, and $\cos\left(\frac{\pi}{4}+x\right)$. These functions were available in the extended primitive table. Meanwhile, for some students, finding primitive functions of $\cos\left(\frac{\pi}{4}+x\right)$ helped them to limit sign errors. In other words, $\int\sin\left(\frac{\pi}{4}-x\right)dx = (-1)\left(-\cos\left(\frac{\pi}{4}-x\right)\right)+C = \cos\left(\frac{\pi}{4}-x\right)+C$ might lead to sign errors, while $\int\cos\left(\frac{\pi}{4}+x\right)dx = \sin\left(\frac{\pi}{4}+x\right)+C$ rarely caused minus sign missing as in the previous function.

The errors that students might deal with when presenting solutions in different ways include these things.

Some of them wrote redundant symbols while transforming function under the integral sign (6 attendants). One example of this error was student 19's paper: $\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4}-x\right)dx = \cos\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-x\right)\right] = \cos\left(\frac{\pi}{4}+x\right)$. In addition, a few students missed the arc sign, or got the incorrect arc of trigonometric functions (4 students). Here was the error of H31: $\frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{2}}\cos x dx + \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{2}} - \sin x dx = \frac{\sqrt{2}}{2}\sin x\Big|_0^{\frac{\pi}{2}} + \frac{\sqrt{2}}{2}\cos\Big|_0^{\frac{\pi}{2}}$. Meanwhile, the right answer should be: $\frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{2}}\cos x dx + \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{2}} - \sin x dx = \frac{\sqrt{2}}{2}\sin x\Big|_0^{\frac{\pi}{2}} + \frac{\sqrt{2}}{2}\cos x\Big|_0^{\frac{\pi}{2}}$. Besides, there are 6 attendants had the wrong sign. Here was the error in H5's task $\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)dx = \int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)d\left(\frac{\pi}{4} - x\right)$, which was supposed to be written as: $\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)dx = -\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)d\left(\frac{\pi}{4} - x\right)$. Another issue was that there were also 2 students having the wrong approach $\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)dx = \int_0^{\frac{\pi}{4}}\cos\left(\frac{\pi}{4} + x\right)dx$ while the correct answer would be: $\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)dx = \int_0^{\frac{\pi}{2}}\cos\left(\frac{\pi}{4} + x\right)dx$.

There are plenty of students (12 students) lacked the upper and lower values after finding primitive $\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)dx = \frac{-\cos\left(\frac{\pi}{4} - x\right)}{\left(\frac{\pi}{4} - x\right)'}$. They should write: $\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)dx = \frac{-\cos\left(\frac{\pi}{4} - x\right)}{\left(\frac{\pi}{4} - x\right)'}\Bigg|_0^{\frac{\pi}{2}}$.

Other students also lacked dx in their paper. Here was the error of one learner in H15 $\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)dx = \int_0^{\frac{\pi}{2}}\cos\left(\frac{\pi}{4} + x\right)$, while the right way is $\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)dx = \int_0^{\frac{\pi}{2}}\cos\left(\frac{\pi}{4} + x\right)dx$. The most serious mistakes belonged to two participants who developed the wrong trigonometric formulas: $\sin(a - b) = \sin a - \sin b$. Meanwhile, there were 6 others who lacked derivative notation sign:

$\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)dx = \frac{-\cos\left(\frac{\pi}{4} - x\right)}{\left(\frac{\pi}{4} - x\right)}\Bigg|_0^{\frac{\pi}{2}}$. These students should write: $\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)dx = \frac{-\cos\left(\frac{\pi}{4} - x\right)}{\left(\frac{\pi}{4} - x\right)'}\Bigg|_0^{\frac{\pi}{2}}$. Some also applied the wrong primitives formula $\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)dx = -\cos\left(\frac{\pi}{4} - x\right)\Big|_0^{\frac{\pi}{2}}$, which should be understood as $\int_0^{\frac{\pi}{2}}\sin\left(\frac{\pi}{4} - x\right)dx = (-1)\left(-\cos\left(\frac{\pi}{4} - x\right)\right)\Big|_0^{\frac{\pi}{2}}$. Finally, only one student did not find the differential when he changed the variables in his paper. Overall, besides the majority of students who had the right solutions differently, the rest ones still had difficulty in the given problem.

Table 3: Statistics of students according to their right solutions

Numbers of solutions	0	1	2	3	4	5
Numbers of students	0 (0%)	0 (0%)	0 (0%)	1 (3.2%)	7 (22.6%)	23 (74.2%)

Table 3 showed that students tried to find out different solutions to the problem. The strength of the experimental group was that they all had 3 or more correct solutions. Among these, 74.2% accounted for 23 students with 5 right solutions. This showed that they had used pretty well the integral knowledge in current textbooks, thereby contributing to the formation of their intellectual qualities. Besides, there were 7 students with 22.6% respectively who had 4 correct methods, and most of them were not successful in using the arc formula. Among 31 experimental attendants, only 15 ones calculated integrals in three ways. The reason was that these students did not solve the problems by transforming trigonometry method.

IV. CONCLUSIONS

The data analysis has pointed out the answers to the research questions: the students were active to solve the problem in different ways. In particular, learners have really exploited the necessary knowledge in textbook to give out reasonable explanation for their solutions, indirectly contributed to the training and improving learners' critical thinking. Another notable conclusion is that there are still a lot of mistakes when students calculate the integral relating to missing dx , lacking upper and lower values, using wrong primitive formulas, wrong values, and not memorizing trigonometric formulas, etc. We believe that this is an interesting and essential research topic which can be further studied in the near future.

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