The International Journal Of Engineering And Science (IJES)
|| Volume || 5 || Issue || 3 || Pages || PP -67- 83 || 2016 ||
ISSN (e): 2319 – 1813 ISSN (p): 2319 – 1805

Operators Smooths Geometric on Riemannian Manifolds

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ABSTRACT
In this paper some fundamental theorems , operators smooth geometry – with operator Riemannian geometry to pervious of differentiable manifolds which are used in an essential way in basic concepts of Spectrum of Discrete , bounded Riemannian geometry, we study the defections, examples of the problem of differentially projection mapping parameterization system on dimensional manifolds **..**

KEYWORDS : *Basic on Riemannian manifolds – Vector analysis one method -Basic on - Dirichlet Problem Function - A smooth Manifold- the Length Minimizing Property of Geodesics-*

Date of Submission: 01 March 2016 Date of Accepted: 20 March 2016

I. INTRODUCTION

The introduction of the paper should explain the nature of the problem, previous work, purpose, and the contribution of the paper. The contents of each section may be provided to understand easily about the paper.

II. BASIC ON RIEMANNIAN MANIFOLDS

2.1: [Topological Manifold]

A topological manifold M of dimension n , is a topological space with the following properties:
(i) M is a Hausdorff space . For ever pair of points $p, g \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $g \in V$. (ii) M is second countable . There exists accountable basis for the topology of M . (c) M is locally Euclidean of dimension n Every point of M has a neighborhood that is homeomorphism to an open subset of R^n .

Definition 2.1.1 [Coordinate Charts]

A coordinate chart or just a chart on a topological $n-$ manifold M is a pair (U, φ) , Where U is an open subset of M and $\varphi : U \to \tilde{U}$ is a homeomorphism from U to an open subset $\tilde{U} = \varphi(U) \subset R^n$.

Examples 2.1.2: [Topological Manifolds Spheres] :

Let S^n denote the (unit) $n-$ sphere, which is the set of unit vectors in R^{n+1}: $S^n = \{x \in R^{n+1} : |x| = 1\}$ with the subspace topology, S^n is a topological $n-$ manifold.

Definition 2.1.3 [Projective spaces]

The $n-$ dimensional real (complex) projective space, denoted by $P_n(R)$ or $P_n(C))$, is defined as the set of 1-dimensional linear subspace of R^{n+1} or C^{n+1}) , $P_n(R)$ or $P_n(C)$ is a topological manifold.

Definition 2.1.4:

For any positive integer n , the $n-$ torus is the product space $T^n = (S^1 \times ... \times S^1)$ .It is an $n-$ dimensional topological manifold. (The 2-torus is usually called simply the torus).

Definition2.1.5 [Boundary of a manifold]

The boundary of a line segment is the two end points; the boundary of a disc is a circle. In general the boundary of an $n-$ manifold is a manifold of dimension $(n-1)$, we denote the boundary of a manifold M as ∂M . The boundary of boundary is always empty, $\partial\partial M = \phi$

Lemma 2.1.6

Every topological manifold has a countable basis of Compact coordinate balls. Every topological manifold is locally compact.

Definitions 2.1.7 [Transition Map]
Let M be a topological space n-manifold. If $(U,\varphi),(V,\psi)$ are two charts such that $U \cap V \neq \phi$, the composite map

(1) $$\psi \circ \varphi^{-1} : \varphi\ (U \cap V) \to \psi\ \ (U \cap V)$$

is called the transition map from φ to ψ .

Definition 2.1.8 [A smooth Atlas]
An atlas A is called a smooth atlas if any two charts in A are smoothly compatible with each other. A smooth atlas A on a topological manifold M is maximal if it is not contained in any strictly larger smooth atlas. (This just means that any chart that is smoothly compatible with every chart in A is already in A.

Definition 2.1.9 [A smooth Structure]
A smooth structure on a topological manifold M is maximal smooth atlas. (Smooth structure are also called differentiable structure or C^{∞} structure by some authors).

Definition 2.1.10 [Lie Algebra]
A Lie algebra is a real vector space g endowed with a map called the bracket from $g \times g\ to\ g$, usually denoted by $(X,Y) \to [X,Y]$, that satisfies the following properties for all $X,Y,Z \in g$:

(i)Linearity: For $a,b \in R$, $[aX + bY, Z] = a[X,Z] + b[Y,Z]$, $[Z, aX + bY] = a[Z,X] + b[Z,Y]$.

(ii) Ant symmetry: $[X,Y] = -[Y,X]$. (iii) Jacobi identity: $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

Example 2.1.11 [Lie Algebra of Vector Fields]
(i) The space (M) of all smooth vector fields on a smooth manifold M is a Lie algebra under the Lie bracket.:

(ii) If G is a Lie group, the set of all smooth left-invariant vector field on G is a Lie sub-algebra of (G) and is therefore a Lie algebra. (iii) The vector space $M(n,R)$ of $n \times n$ matrices an n^2 – dimensional Lie algebra under the commentator bracket: $[A,B] = AB - BA$. Linearity and ant symmetry are odious from the definition, and the Jacobi identity follows from a straight forward calculation. When we are regarding $M(n,R)$ as a Lie algebra with this bracket, we will denote it by $gl\ (n,R)$.

Definition 2.1.12 [A smooth Manifold]
A smooth manifold is a pair $(M$, A), where M is a topological manifold and A is smooth structure on M . When the smooth structure is understood, we omit mention of it and just say M is a smooth manifold.

Definition 2.1.13
Let M be a topological manifold. (i) Every smooth atlases for M is contained in a unique maximal smooth atlas.(ii) Two smooth atlases for M determine the same maximal smooth atlas if and only if their union is smooth atlas.

Definition 2.1.14
Every smooth manifold has a countable basis of pre-compact smooth coordinate balls. For example the General Linear Group The general linear group $GL\ (n,R)$ is the set of invertible $n \times n$ -matrices with real entries. It is a smooth n^2 -dimensional manifold because it is an open subset of the n^2 - dimensional vector space $M(n,R)$, namely the set where the (continuous) determinant function is nonzero.

Definition 2.1.15 [Tangent Vectors on A manifold]
Let M be a smooth manifold and let p be a point of M . A linear map $X : C^{\infty}(M) \to R$ is called a derivation at p if it satisfies:

(2) $$X(fg) = f(p)Xg + g(p)Xf$$

For all $f,g \in C^{\infty}(M)$. The set of all derivation of $C^{\infty}(M)$ at p is vector space called the tangent space to M at p , and is denoted by [T_pM]. An element of T_pM is called a tangent vector at p .

Lemma 2.1.16 [Properties of Tangent Vectors.]
Let M be a smooth manifold, and suppose $p \in M$ and $X \in T_pM$. If f is a const and function, then $(Xf) = 0$. If $f(p) = g(p) = 0$, then $X(fp) = 0$.

Definition2.1.17 [Tangent Vectors to Smooth Curves]
If γ is a smooth curve (a continuous map $\gamma : (J \to M)$,where $J \subset R$ is an interval) in a smooth manifold M , we define the tangent vector to γ at $t_{\circ} \in J$ to be the vector

(3) $$\gamma'(t_{\circ}) = \gamma_{*}\left(\frac{d}{dt}|_{t_{\circ}}\right) \in T_{\gamma(t_{\circ})}M$$, where $d/dt\ |_{t_{\circ}}$ is the

standard coordinate basis for $T_{t_\circ} R$. Other common notations for the tangent vector to γ are $\left[\gamma^*(t_\circ), \frac{d\gamma}{dt}(t_\circ)\right]$ and $\left[\frac{d\gamma}{dt}|_{t=t_\circ}\right]$. This tangent vector acts on functions by:

(4) $$\left[\gamma'(t_\circ)\ f = \left(\gamma_* \frac{d}{dt}|_{t_\circ}\right) f = \frac{d}{dt}|_{t_\circ}(f \circ \gamma) = \frac{d(f \circ \gamma)}{dt}(t_\circ)\right].$$

Lemma 2.1.18 [Smooth manifold]
Let M be a smooth manifold and $p \in M$. Every $X \in (T_p M)$ is the tangent vector to some smooth curve in M .

2.3 : VECTOR ANLYSIS ONE METHOD LENGTHS

Classical vector analysis describes one method of measuring lengths of smooth paths in $\hat{R}$ if $v:[0,1] \to \hat{R}$ is such a paths, then its length is given by $v = \int |v(t)|\, dt$ Length. Where $|v|$ is the Euclidean length of the tangent vector (t) , we want to do the same thing on an abstract manifold, and we are clearly faced with one problem, how do we make sense of the length $|v(t)|$, obviously , this problem can be solved if we assume that there is a procedure of measuring lengths of tangent vectors at any point on our manifold . The simplest way to do achieve this is to assume that each tangent space is endowed with an inner product. (Which can vary point in a smooth).

Definition 2.3.1
A Riemannian manifold is a pair (M, g) consisting of a smooth manifold M and a metric g on the tangent bundle,(i.e.) a smooth symmetric positive definite tensor field on M . The tensor g is called a Riemannian metric on M . Two Riemannian manifold are said to be isometric if there exists a diffoemorphism $\varphi : M_1 \to M_2$ such that $\hat{\varphi}^*, g_2 = g_1$. If (M, g) is a Riemannian manifold then, for any $x \in M$ the restriction $g_x : T_x \times T_x M \to R$
Is an inner product on the tangent space $T_x M$ we will frequently use thee alternative notation $[\cdot,\cdot], x = g_x\ [\cdot,\cdot]$ the length of a tangent vector $v \in T_x M$ is defined as usual $|v|, x = g_x[v, v] \wedge 1/2$ If $v:[a,b] \to M$ is a piecewise smooth path , then we defined is length by $L(v) = \int a \wedge b\, |v(t)| dt$. If we choose local coordinates $(x^1, ..., x^n)$ on M then we get a local description of g as $g = g_{ij}(dx^1{}_i \wedge dx^2_i)$

Proposition 2.3.2
Let be a smooth manifold , and denote by R_M the set of Riemannian metrics on M ,then R_M is a non –empty convex cone in the linear of symmetric – tensor
Proof:
The only thing that is not obvious is that R_M is non-empty we will use again partitions of unity. Cover M by coordinate neighborhoods $(U, \alpha)\alpha \in A$. Let (x_i^α) be a collection of local coordinates on U_α . Using these local coordinates we can construct by hand the metric g_α on U_α by $g_\alpha = (dx_\alpha^1)^2 + ... + (dx_\alpha^n)^2$. Now, pick a partition of unity $B \subset C_0^\infty(M)$. Subordinated to cover $(U, \alpha)\alpha \in A$. (i.e) there exists a map $\varphi : B \to A$ such that $\forall \beta \in B \subset U, \varphi \in B$ then define $g = \sum \beta \in B\ [\beta g \varphi(B)]$. The reader can check easily g is well defined, and it is indeed a Riemann metric on M .

Example 2.3.3 [The Euclidean Space]
The space R^n has a natural Riemann metric $g_0 = (dx_\alpha^1)^2 + ... + (dx_\alpha^n)^2$
The geometry of R_g^n is the classical Euclidean geometry

Example2.3.4 [Induced Metrics on Sub manifolds]
Let $(M.g)$ be Riemann manifold and $S \subset M$ a sub manifold if: $S \to M$, denotes the natural inclusion then we obtain by pull back a metric on S $g_s{}^i = g = g_s$. For example, any invertible symmetric $n \times n$ matrix defines a quadratic hyper surface in R^n by $H_A = \{x \in R^n : (Ax, x) = 1\}$ where $[\cdot,\cdot]$ denotes the Euclidean inner on R^n , H_A has a natural

Remark 2.3.5
On any manifold there exist many Riemannian metrics, and there is not natural way of selecting on of them. One can visualize a Riemannian structure as defining "shape" of the manifold. For example, the unit sphere $(x^2 + y^2 + z^2) = 1$, is diffeomorphic to the ellipsoid $(x^{1/2})^2 + (y^{1/2})^2 + (z^{1/2})^2 = 1$.,but they look "different" by. However, appearances may be deceiving in is illustrated the deformation of a cylinder they look different, but

the metric structures are the same since we have not change length of curves on our sheep. The conclusion to be drawn from these two examples is that we have to be very careful when we use the attribute "different".

3.1.6 Example:2.3.6 [The Hyperbolic Plane]
The Poincare model of the hyperbolic plane is the Riemannian manifold (D, g) where D is the unit open disk in the plan R^2 and the metric g is given by $g = 1/(1 - x^2 - y^2)(dx^2 + dy^2 + dz^2)$

Example2.3.7 [Left Invariant Metrics on lie groups]
Consider a lie group G ,and denote by L_G its lie algebra then any inner product $\langle .,. \rangle$ on L_G induces a Riemannian metric $h = \langle .,. \rangle_g$ on G defined by $h_g(X, Y) = \langle x, y \rangle, g = (L_g^{-1}) * X, (L_g^{-1}) * Y \quad , \forall g \in G, X, x \in T$

Where $[L_g^{-1}]_* : T_g G \to T_1 G$ is the differential at $g \in G$ of the left translation map L_g^{-1}. One checks easily that check easily that the correspondence $G \in g \to \langle .,. \rangle$ is a smooth tensor field, and it is left invariant (i,e) $L^* h = h$ If G is also compact ,we can use the averaging technician to produce metrics which are both left and right invariant .

2.4 : The Levi-Cavite Connection]
To continue our study of Riemannian manifolds we will try to follow a close parallel with classical Euclidean geometry the first question one may ask is whether there is a notion of "straight line" on a Riemannian manifold .In the Euclidean space R^3 there are at least ways to define a line segment. A line segment is the shortest path connecting two given points. A line segment is a smooth path $v : [0,1] \to R^3$ satisfying $\ddot{v}(t) = 0$. Since we have not said anything about calculus of variations which deals precisely with problems of type (i) we will use the second interpretation as our starting point, we will soon see however that both points of view yield the same conclusion. Let us first reformulate. As know the tangent bundle of R^3 is equipped with a natural trivialization, and as such it has a natural trivial connection ∇_0 defined by $\nabla_0^i \partial_j = 0, \ \forall_{i,j}$ where $\partial_i = \partial(\partial_x^i), \nabla_i = \nabla_{\partial_i}$ (i.e) all the christoffel symbols vanish, moreover, if g_0 denotes the Euclidean metric, then .

$$(\nabla_0^1, g_0)(\partial_1^k) = \nabla_i \delta_{jk} = v_i^0 \delta_{jk}^0 \ (\nabla_i \partial_j \partial_k) - g_0 \circ (\partial_j, \nabla_0 \partial_k) = 0$$

$\nabla_{v(t)} v(t) = 0$. So that the problem of defining " lines " in a Riemannian manifold reduces to choosing a " natural " connection on the tangent bundle of course , we would like this connection to be compatible with the metric but even so , there infinitely many connections to choose from . The following fundamental result will solve this dilemma.

Proposition 2.4.1 [Levi-Cavite Connection]
Consider a Riemannian manifold $(M.g)$,then there exists a unique symmetric connection ∇ on TM compatible with the metric g, (i,e) $T(\nabla) = 0, \nabla_g = 0$ the connection ∇ is usually called the Levi-civet connection associated to the metric g .

Proof
Uniqueness we will achieve this by producing an explicit description of a connection with the above two m properties let ∇ be such a connection, (i.e) $\nabla_g = 0$ and $\nabla_x Y - \nabla_y X = [X, Y], \forall : X, Y \in (M)$ for any we have, $Z_g(X, Y) = Z_g(X, Y) = g(\nabla_z X, Y) + g(X \nabla_z Y)$ since.

$(X, Y) - Y_g(Z, X) + X_g(Y, Z) = g(V_z X, Y) - g(V_y Z, X) + g(\nabla_x Y, Z) + g(X, \nabla_z, Y) - g(Z, \nabla_y X) + g(Y, \nabla_y X) + g(Y, VX, Z)$

We conclude that $g(\nabla_z X, Y) = 1/2 \quad \{X_g(Y, Z) - Y_g(Z, X) + Z_g\}(X, Y) - g([X, Y], Y) + g([Y, Z], X) - g([Z, X], Y)$

The above equality establishes the uniqueness of ∇ using local coordinates $x^1, ..., x^n$ on M we deduce from with .

(5)
$$X = \partial_i = \partial / (\partial_{Xi}), Y = \partial_k = \partial / (\partial_{Xk}), \ Z = \partial / (\partial_{Xj})$$

$$(\nabla_i \partial_j \partial_k) = g_{ki} \ \Gamma_{ki}^1 = 1/2(\partial_i g_{ki} - \partial_i g_{ki} - \partial_k g_{ki})$$

Above ,the scalars Γ_{ij}^1 denote the Christ symbols of ∇ in these coordinates ,(i.e) $\nabla_{\partial i} \partial_j = \Gamma_{ij}^1 \partial_i$ If $g^1{}_k$ denotes the inverse of g_i^1 we deduce $\Gamma_{ij}^1 = 1/2 \, g_{ki}^1 (\partial_i g_{ik} - \partial_k g_{ij} + \partial_j g_{ik})$

Definition 2.4.2 :[Riemannian Manifold Is Smooth]
A geodesic on a Riemannian manifold (M, g) is a smooth path, $v : (a, b) \to M$, satisfying $\nabla v(t), v(t) = 0$

Where ∇ is the "Levi - Cavite " connection. Using local coordinates $x^1, ..., x^n$ with respect to which the Christoffel simples are Γ_{ij}^k and the path v is described $v(t) = x^1(t), ..., x^n(t)$ we can rewrite the geodesic equation as a second, order nonlinear system of ordinary differential equations set $d/dt = v(t) = x^i \partial_i$,

$$\nabla(d/dt)v(t) = \ddot{x}\partial_i + \ddot{x}_j\nabla(d/dt) \quad , \quad \partial_i = \ddot{x}\partial_i + \ddot{x}^i x^j \nabla_j \partial_i = \ddot{x}^k\partial_k + \Gamma^k_{ij}\dot{x} + \Gamma^k_{ij}x\left(\Gamma^k_{ij} = \Gamma^k_{ij}\right)$$

So the geodesic equation is equivalent to $x^k + \Gamma^k_{ij} \times x^j = 0$, $\forall : k = 1,...,n$. Since the coefficients $\Gamma^k_{ij} = \Gamma^k_{ij}(x)$ depend smooth up on , x we can use the classical Banish-Picard

Proposition 2.4.3 [Riemannian for any Compact subset]

Let (M,g) be a Riemannian manifold for any compact subset $\subset TM$, there exists $\varepsilon \geq 0$ such that for any $(x, X) \in k$ there exists a unique geodesic $v = v_X$, $X = (-\varepsilon, \varepsilon) \to M$ such that $v(0) = x$, $v(0) = X$. One can think of a geodesic as defining a path in the tangent bundle $t \to (v(t), v(t)$, $v = v_X X$. The above proposition shows that the geodesics define a local flow φ on (TM) by $\varphi^t(x, X) = v(t), v(t)$, $v = v_X X$.

Definition 2.4.4 :[Geodesic Low]

The local flow defined above is called the geodesic flow the Riemannian manifold (M, g). When the geodesic low is global flow , (i.e) any $v_x X$ is defined at each moment of t for any $(x, X) \in TM$, then the Riemannian manifold is call geodetically complete .

Proposition 2.4.5 : [Conservation of energy]

If the path $v(t)$ is a geodesic , then length of $v(t)$ is independent of time

Proof :

we have $d/dt \left| v(t) \right|^2 = d/dt \; g(v(t), v(t) = 2g(\nabla - v(t), v(t)) = 0$. Thus, if we consider the sphere bundles $S_r(M) = \{x \in TM , |x| = r\}$. We deduce that $S_r(M)$ are invariant subset of geodesic flow .

Definition 2.4.6 :[Lie algebra Group]

Let G be a connected lie group, and let L_G be its lie algebra. Any $x \in L_G$, defines an endomorphism $ad(X)$ of L_G by $ad(X)Y = [X, Y]$ The Jacobi identity implies that $ad(X)Y = [ad(X), ad(Y)]$ where the bracket in the right hand side is the usual commentator of two endomorphism. Assume that there exists an inner product $\langle .,. \rangle$ on L_G such that for any $x \in L_G$ the operator $ad(X)$ is skew-adjoin (i.e) $\langle [X, Y], Z \rangle = \langle Y, [X, Y] \rangle$. We can now extend this inner product to a left invariant metric h on G . We want to describe its geodesic first, we have to determine associated "Levi-civet" connection .using we get .

(6) $$h(\nabla_X Z, Y) = 1/2, \{Xh(Y, Z) - Y(Z, X) + Zh(X, Y) - h([Y, Z] - h([Z, X], Y\}$$

If we take $X, Y, Z \in L_G$ (i.e) these vector fields are left invariant ,the $h(Y, Z) = const$, $h(Z, X) = const$, $h(X, Y) = const$, so that the first three terms in the above formula vanish we obtain the following equality at $1 \in G$.

(7) $$\langle \nabla_x Z, Y \rangle = 1/2 , \{- \langle [X, Z], Z \rangle + \langle -[Y, Z], X \rangle + \langle -[Z, X], X \rangle\}$$

Using the skew-symmetry of $ad(X)$ and $ad(Z)$ we deduce $\langle \nabla_x Z, Y \rangle = 1/2$, $\langle [X, Z], Y \rangle$ so that.

(8) $$1 \in G \quad , \quad \nabla_x Z = 1/2[X, Y] \quad \forall : X, Z \in L_G$$

This formula correctly defines a connection since any $X \in vector \; (G)$ can be written as a linear combination.

(9) $$X = \sum[\alpha_i X_i] \; , \; \alpha_i \in C^\infty(G) \; , \; X_i \in L_G$$

If $v(t)$ is a geodesic ,we can write $v(t) = \sum[v_i X_i]$, so that $\nabla \;\; = 0 \;\; t = v_i X_i + 1/2 \sum_{ij} v_i v_j [X_i, X_j]$

Since $[X_i, X_j] = -[X_j, X_i]$ we deduce $v_i = 0$, $v(t) = \sum[v_i(0), X_i = X]$. This means that v is an integral curve of the left invariant vector field X ,so that the geodesics through the origin with initial direction $X \in T_G$ are $v_x(t)) = \exp \; f(t)_x$

Definition 2.4.7 : [Killing Paring]

Let L be a finite dimensional real lie algebra ,the killing paring or form is the bilinear map

$$K : L \times L \to R \quad , k(x, y) = -tr\,(ad(x)\,ad(y) \quad , \forall : x, y \in L$$

The lie algebra L is said to be semi simple if killing paring is a duality .A lie group G is called semi simple if its lie algebra is semi simple .

2.5 : The Exponential Map Normal Coordinates

We have already seen that there are many difference between the classical Euclidean geometry and the general Riemannian geometry in the large. In particular we have seen examples in which one of basic axioms of Euclidean geometry no longer holds .Two distinct geodesic (real lines) may intersect in more than one point. The global topology of the manifold is responsible for this "failure" . In this we will define using the metric some special collections to being Euclidean. Let (M, g) be Riemannian manifold and u, an open coordinate neighborhood with coordinate $x^1,....,x^n$ .We will try to find a local change in coordinate $x^i \to y^i$ in which the expression of the metric is as close are to the Euclidean metric $g_0 = \delta_{ij}$, dy^i, dy^j . Let $q \in u$,be the point with coordinate $0,....,0$ via a linear we may as well assume that $g_{ij}(q)=\delta_{ij}$ .We would like "spread" the above equality to an entire neighborhood of q . To achieve this we try to find local coordinates y^j near q such that in these new coordinates the metric is Euclidean up to order one (i,e)

$$g_{ij}(q)=\delta_{ij}\ ,(\partial g_{ij})/(\partial y^k)(q)=(\partial\delta_{ij})/(\partial y^k)/(q)=0 \quad , \ \forall\ ;i,\ j\in k$$

We now describe a geometric way of producing such coordinates using the geodesic flow .Denote as usual the geodesic from q with initial direction $X\in T_qM$ by $\gamma_q X(t)$ Not the following simple fact $\forall S\geq 0\ ,\gamma_q X(t)=\gamma_q X(St)$ ∀s>0 . Hence , there exists a small neighborhood V of T_qM , Such that , for any $X\in V$,the geodesic $\gamma_q X(t)$ is defined for all $|t|\leq 1$ .we define the exponential map at q .

$\exp\ q:V\subset T\ ,q:M\rightarrow M\ ,X\rightarrow\gamma_q X(1)$

The tangent space T_qM is a Euclidean space , and we can define $D_q(r)\subset T_qM$, the open "disk" of radius r centered at the origin we have the following result centered at the origin .we have the following result

Proposition 2.5.1: [Radii]

Let (M,g) and $q\in M$ as above .Then there exists $r\geq 0$ such that the exponential map

$$\exp\ q:D_q(r)\rightarrow M$$

Is a diffoemorphism on to .The supermom of all radii r with this property is denoted $P_M(q)$.

Definition 2.5.2 : [Infectivity Radius of M]

The positive real number $P_M(q)$ is called the infectivity radius of M at q .the infimum

$$P_M=\inf\ _qP_M(q)$$

Is called the infectivity radius of M

Lemma2.5.3: [Freshet Differential]

The Freshet differential at $0\in T_qM$ of the exponential map.

$$D_0\exp\ q:T_qM\rightarrow T\ ,\exp\ q(0)M=T_qM$$

Is the identity $T_qM\rightarrow T_qM$

Proposition 2.5.4: [Metric Tensor]

Let x^i be normal coordinates at $\in M$, and denote by g_{ij} , the expression of the metric tensor in these coordinates then we have $g_{ij}(q)=\delta_{ij}$ and $\partial\delta_{ij}/\partial x^k(q)=0\ \forall :i\ j\in k$

Thus, the normal coordinates provide a first order contact between g, and the Euclidean metric.

Lemma 2.5.5:

In normal coordinates x^i at q the christoffel symbols Γ^{ik} vanish at q

2.6 : [The Length Minimizing Property Of Geodesics]

For each $q\in M$, there exists $0\leq r\leq P_M(q)$ and $\varepsilon\geq 0$ such that $\forall:m\in B_r(q)$, we have $\varepsilon\leq p_mM$ and $B\subset(m)\subset B_r(q)$ in particular , any two of $B_r(q)$ can be joined by a unique geodesic of length ε . We must warn the reader the above result does not guarantee that the postulated connecting geodesic lies entirely in $B_r(q)$ .This is a different ball game .

Theorem 2.6.1 : [Unique Geodesic]

Let q,r and ε as in the previous and consider the unique geodesic $r:[0,1]\rightarrow M$ of length <ε, joining two points of $B_r(q)$ if $w:[0,1]\rightarrow M$ is a a piecewise smooth path with the same endpoint as γ then.

$$\int|\gamma(t)|dt\leq|w(t)|dt$$

With equality if and only if $w:[0,1]=\gamma[0,1]$ .Thus γ is the shortest path, joining its endpoints.

2.7 : Riemannian Geometry

Definition 2.7.1: [Riemannian Metrics]

Differential forms and the exterior derivative provide one piece of analysis on manifolds which, as we have seen, links in with global topological questions. There is much more on can do when on introduces a Riemannian metric. Since the whole subject of Riemannian geometry is a huge to the use of differential forms. The study of harmonic from and of geodesics in particular, we ignore completely hare questions related to curvature.

Definition 2.7.2: [Metric Tensor]

In informal terms a Riemannian metric on a manifold M M is a smooth varying positive definite inner product on tangent space. To make global sense of this note that an inner product is a bilinear form so at each point we want a vector in tensor product. We can put, just as we did for exterior forms a vector bundle striation on

The conditions we need to satisfy for a vector bundle are provided two facts we used for the bundle of p-forms each coordinate system defiance a basis for each in the coordinate neighborhood and the element .Given a corresponding basis for. The Jacobean of a change of coordinates defines an invertible linear transformation. And we have a corresponding.

Definition 2.7.3 : [Local Coordinate System]

A Riemannian metric on manifold M is a section g of which at each point is symmetric and positive definite. In a local coordinate system we can write . Where and is a smooth function, with positive definite. Often the tensor product symbol is omitted and one simply writes.

Definition 2.7.4 :[Two Riemannian Manifold Is an Isometric]

A diffoemorphism, between two Riemannian manifold is an isometric if

Definition 2.7.5: [Upper half-plan]

Let , and , if and then

S0 these Movies transformation are isometrics of Riemannian metric on the upper half-plan.

Definition 2.7.6:[Smooth Curve in M]

Let M be a Riemannian manifold and a smooth map a smooth curve in M . The length of curve is with and , then . So these Movies transformation are isometrics of Riemannian metric on the upper half-plan.

Definition 2.7.7:[A smooth Curve]

Let M a Riemannian manifold and a smooth map I,e a smooth curve in M . The length of curve is

Where , with this definition, any Riemannian manifold is metric space define are Riemannian and manifold space.

Proposition 2.7.8:[Manifold admits a Riemannian Metris]

Any manifold a demits a Riemannian metric

Proof :

Take a converging by coordinate neighborhoods and a partition of unit subordinate to covering. On each open set we have a metric. In the local coordinates, define this sum is well-defined because the support of. Are locally finite. Since at each point every term in the sum is positive definite or zero, but at least one is positive definite so that sum is positive definite.

Definition 2.7.9 : [The Geodesic Flow]

Consider any manifold M and its cotangent bundle , with projection to the base , let X be tangent vector to at the point then so that Defines a conical 1-form on in coordinates the projection P is so if so if given take the exterior derivative which is the canonical 2-from on the cotangent bundle it is non-degenerate, so that the map from the tangent bundle of to its contingent bundle is isomorphism. Now suppose f is smooth function an its derivative is a 1-form df d .Because of the isomorphism a above there is a unique vector field X on such that from the g another function with vector field Y , then. $Y(f) = def\ .(y)$ On a Riemannian manifold we shall see next there is natural function on . In fact a metric defines an inner on as well as on T for the map defines an isomorphism form T to then which means that where denotes the matrix .

Definition2.7.10: [Geodesist Metric]

The vector field X on given by is called the geodesist flow of the metric g .

Proposition2.7.11: [Projects Riemannian Manifold]

The function f a above is If Write in coordinates If where If since projects on M then by the definition of .Now let M be a Riemannian manifold and H , the function on defined by the metric as a above , if is an one parameter group of isometrics , then the induced diffoemorphism of will preserve the function H so the vector field will satisfy . that where X is the geodesic flow along the geodesic flow, and is therefore a constant of integration of the geodesic equations

Theorem 2.7.12 [Harmonic Function is maximum and minimum]

Suppose that Ω is a connected open set and $U \in C^2(\Omega)$, if U harmonic and attains a global minimum or maximum in Ω .then U is constant

Proof:

Any super harmonic function U that attains minimum Ω is constant since, $-U$ is sup harmonic and attains a maximum a harmonic function is both sub harmonic.

Example 2.7.13

The function $U(x, y) = x^2 - y^2$ is harmonic in R^n it's the real part of the analytic function $f(z) = z^2$ it has critical point at 0 meaning that $D_u = 0$,this critical point is a saddle –point however not an extreme value not also that.

(1) $$\int_{\beta_{r(0)}} U\ dx\ dy = \frac{1}{2\pi}\int_0^{2\pi}(\cos^2\phi - \sin^2\phi)\,d\phi = 0$$

as required by mean value property , one consequence of this property is that any non-constant harmonic function is an open mapping meaning , that it maps opens set to open set this not true of smooth function such as $x \to |x|^2$ that. Extreme value

Theorem 2.7.14
Suppose that Ω is a bounded, connected open set in R^n and $U \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω then.

(2) $$(\max_{\Omega} U) = (\max_{\partial\Omega} U) \ (\min_{\Omega} U) = (\min_{\partial\Omega} U)$$

Proof:
Since U is continuous and $\overline{\Omega}$ is compact , U attain its global maximum and minimum on $\overline{\Omega}$, if U attains a maximum or minimum value at interior point then U is constant by otherwise both extreme values are attained in the boundary .In either cases the result follows let given a second of this theorem that does not depend on the mean value property .Instated we us argument based on the non-positivity of the second derivative at an interior maximum . In the proof we need to account for the possibility of degenerate maxima where the second derivative in zero. For $\varepsilon \geq 0,\ let\ U^{\varepsilon}(x) = U(x) + \varepsilon |x|^2$. Then $\Delta U^{\varepsilon} = 2n \quad \varepsilon \geq 0$, since U is harmonic .if U^{ε} attained a local maximum at an interior point then $\Delta U^{\varepsilon} \leq 0$ by the second derivative test. Thus U^{ε} no interior maximum, and it attains its maximum on the boundary .If, $|x| \leq R,\ for\ all\ x \in \Omega$, if follows that.

(4) $$(Sup_{\Omega} U \leq Sup_{\Omega} U^{\varepsilon}) \leq Sup_{\partial\Omega} U^{\varepsilon} \leq (Sup_{\partial\Omega} U + \varepsilon R^2)$$

Letting $\varepsilon \to 0^+$,we get that $(Sup_{\Omega} U) \leq (Sup_{\partial\Omega} U)$ .An application for the same a grummet to u given in, $\inf_{\Omega} U \leq \inf_{\partial\Omega} U$ .and the result follows . Sub harmonic function satisfy a maximum principle $(\min_{\overline{\Omega}} U) = (\min_{\partial\Omega} U)$,while sub harmonic function satisfy a minimum principle $(\min_{\overline{\Omega}} U) \leq U \leq (\min_{\partial\Omega} U)$ for all $x \in \Omega$. Physical terms, this means for example that the interior of abounded region which contains no heat sources on heat sources or sinks cannot be hotter that the maximum temperature on the boundary or colder than the minimum temperature on the boundary .The maximum principle given a uniqueness result for (Dirichlet problem) for the poison equation .

Theorem 2.7.15 [Dirichlet Problem Function]
Suppose that Ω is a bounded connected open set in R^n and $f \in C(\Omega), g \notin (\partial\Omega)$ are given function then is at most one solution of the" Dirichlet problem" with $U \in C^2(\Omega) \cap C(\overline{\Omega})$.

Proof:
Suppose that $U_1, U_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy equation $(-\Delta U = f\ in\ \Omega), (U = g\ on\ \partial\Omega)$, Let $V = U_2$ then, $V \in (C^2(\Omega) \cap C(\overline{\Omega}))$ is harmonic Ω and, $V = 0\ on\ \partial\Omega$,the maximum principle implies that $V = 0\ in\ \Omega$ so $U_1 = U_2$ and solution is unique .

Definition 2.7.16 [**The Maximum Principle and Uniqueness**]
Are our solution formulas the only solution of the heat equation with the specified initial and or boundary condition by linearity. This amounts to asking whether $U = 0$ is the only date 0 the answer is yes . We shall prove this using the (maximum principle) .The maximum principle this is an elementary for - reaching fact about solutions of linear parabolic equation. Here is the simplest version. Let D be the bounded domain suppose $f_t - \Delta f \leq 0, \forall x \in D$ and $0 \leq t \leq T$ Then the maximum of f in the maximum closed cylinder $\overline{D} \times [0, T]$ is a chive either at the (initial boundary) $t = 0$ at the (spatial boundary) ${}^r(V) \supset \Lambda^r(V)$. Notice the asymmetry between the initial boundary $t = 0$, (where f can easily easily a chive its maximum) and the finial boundary ϕ (where f does not achieve its maximum except in trivial case when f is constant) . This asymmetry reflects once again time has (preferred $\varphi(v_1, ..., v_n)$ when solving a parabolic P. D .E)

III. OPECTIONS TENSOR FIELDS AND REMANNIAN MANFOLDS

3.1 Tensor Fields

Definition 3.1.1 [a convector Tensor.]
A convector tensor on a vector space V is simply a real valued $\phi(v_1, ..., v_r)$ of several vector variables $v_1, ..., v_r$ of V , linear in each separately.(i.e. multiline). The number of variables is called the order of the tensor. A tensor field ϕ of order r on a manifold M is an assignment to each point $P \in M$ of a tensor ϕ_P on the vector space $T_P(M)$, which satisfies a suitable regularity condition C^0, C^r, orC^{∞} as P varies on M .

Theorem 3.1.2
With the natural definitions of addition and multiplication by elements of R the set $(V)^r_s$ of all tensors of order (r, s) on V forms a vector space of dimension n^{r+s} .

Definition 3.1.3 [Tensor Fields.]
A C^{∞} covariant tensor field of order r on a C^{∞} - manifold M is a function ϕ which assigns to each $P \in M$ an element ϕ_p of $(T_P(M))^r$ and which has the additional property that given any C^{∞} – Vector fields on an open subset U of M, then $\phi(X_1,..., X_r)$ is a C^{∞} function on U, defined by, $\phi(X_1,..., X_r)\ (P) = \phi_P(X_{1P},...., X_{rP})$. We denote by $(M)^r$ the set of all C^{∞} – covariant tensor fields of order r on M.

Definition 3.1.4
We shall say that $\phi \in V^r$, $\phi \in V^r$ a vector space, is symmetric if for each $1 \le i, j \le r$, we have : $\phi(v_1,..., v_j,..., v_i,..., v_r) = \phi(v_r,..., v_j,..., v_i,..., v_1)$ .Similarly, if interchanging the (i-th) and (j-th) variables, $1 \le i, j \le r$ Changes the sign, $\phi(v_1,..., v_j,..., v_i,..., v_r) = -\phi(v_r,..., v_j,..., v_i,..., v_1)$,then we say ϕ is skew or anti symmetric or alternating; covariant tensors are often called exterior forms. A tensor field is symmetric (respectively, alternating) if it has this property at each point.

Theorem 3.1.5
Let $F : M \to N$ be a C^{∞} map of C^{∞} manifolds. Then each C^{∞} covariant tensor field ϕ on N determines a C^{∞} covariant tensor field $F^*\phi$ on M by the formula $(F^*\phi)_P(X_{1P},..., X_{rP}) = \phi_{F(P)}(F^*(X_{1P}),..., F^*(X_{rP}))$. The map $F^* : {}^r(N) \to {}^r(M)$ so defined is linear and takes symmetric (alternating) tenors to symmetric (alternating) tensors.

Theorem 3.1.6
The maps A and S are defined on $(M)^r$ a C^{∞}- manifold and $(M)^r$ the C^{∞}- covariant tensor fields of order r, and they satisfy properties there. In these case of, $F^* : {}^r(N) \to {}^r(M)$ is the linear map induced by a C^{∞} mapping $F : M \to N$.

Definition3.1.7 [Multiplication of Tensors on Vector Space].
Let V be a vector space and $\varphi \in V$ are tensors. The product of φ and ψ, denoted $\varphi \otimes \psi$ is a tensor of order $r + s$ defined by $\varphi \otimes \psi\,(v_1,...v_r,...v_{r+1},....v_{r+s}) = \varphi(v_1,...., v_r)\psi(v_{r+1},...., v_{r+s})$.
The right hand side is the product of the values of φ and ψ .The product defines a mapping $(\varphi, \psi) \to \varphi \otimes \psi$ of $\mathrm{x}\,{}^r(V) \to {}^{r+s}(V)$.

Theorem 3.1.8
The product ${}^r(V)\ \mathrm{o}\ {}^r(V) \to {}^{r+s}(V)$ just defined is bilinear and associative. If $\omega^1,...., \omega^n$ is a basis of .

(5) $$V^* = \varphi_1 \wedge \varphi_2 \wedge .... \wedge \varphi_k = \left(\frac{(r_1 + r_2 + ... + r_k)!}{r_1!\, r_2!.... r_k!} \right)$$

then $\{(\omega^{i_1} \otimes .... \otimes \omega^{i_r}) / (1 \le i_1,..., i_r \le n)\}$ is a basis of $(V)^r$. Finally $F_* : W \to V$ is linear, then $F^*(\varphi \otimes \psi) = (F^*\varphi) \otimes (F^*\psi)$.

Proof:
Each statement is proved by straightforward computation. To say that $\otimes$ is bilinear means that if α, β are numbers $\varphi_1, \varphi_2, \in (V)^r$ and $\psi \in {}^r(V)$, then $(\alpha\varphi_1 + \beta\varphi_2) \otimes \psi = \alpha(\varphi_1 \otimes \psi) + \beta(\varphi_2 \otimes \psi)$. Similarly for the second variable. This is checked by evaluating each Side on $r + s$ vectors of V ; in fact basis vectors suffice because of linearity Associatively, $(\varphi \otimes \psi) \otimes \theta = \varphi \otimes (\psi \otimes \theta)$,is similarly verified the products on both sides being defined in the natural way. This allows us to drop the parentheses. To see that $\omega^{i_1} \otimes ..... \otimes \omega^{i_r}$ from a basis it is sufficient to note that if $e_1,..., e_n$ is the basis of V dual to $\omega^1,...., \omega^n$, then the tensor $\Omega^{i_1,...,i_r}$ previously defined is exactly $\omega^{i_1} \otimes ..... \otimes \omega^{i_r}$ .This follows from the two definitions:

(6) $$\Omega^{i_1...i_r}(e_{j_1},..., e_{j_r}) = \begin{cases} 0 & if\ (i_1,..., i_r) \ne (j_1,..., j_r) \\ 1 & if\ (i_1,..., i_r) = (j_1,..., j_r) \end{cases},$$

and $(\omega^{i_1} \otimes .... \otimes \omega^{i_r})\ (e_{j_1},..., e_{j_r}) = \omega^{i_1}(e_{j_1})\omega^{i_2}(e_{j_2}),...,\ \omega^{i_r}(e_{j_r}) = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}...\delta^{i_r}_{j_r}$,which show that both tensors have the same values on any (ordered) set of r basis vectors and are thus equal. Finally, given $F_* : W \to V$, if $w_1,..., w_{r+s} \in W$, then
$(F^*(\varphi \otimes \psi))\,(w_1,..., w_{r+s}) = \varphi \otimes \psi\ \ (F_*(w_1),..., F_*(w_{r+s})) = \varphi(F_*(w_1),..., F_*(w_r))\ \psi(F_*(w_{r+1}),...., F_*(w_{r+s}))$
$= (F^*\varphi) \otimes (F^*\psi)(w_1,....., w_{r+s})$.
Which proves $F^*(\varphi \otimes \psi) = (F^*\varphi) \otimes (F^*\psi)$ and completes the proof.

Theorem 3.1.9 : [Multiplication of Tensor Field on Manifold]
Let the mapping $(M)^r \times (M)^s \to (M)^{r+s}$ just defined is bilinear and associative. If $(\omega^1,....,\omega^n)$ is a basis of $(M)^1$, then every element $(M)^r$ is a linear combination with C^∞ coefficients of $\{(\omega^{i_1} \otimes .... \otimes \omega^{i_r})/(1 \le i_1,..., i_r \le n)\}$. If $F : N \to M$ is a C^∞ mapping, $\varphi \in M$ and $\psi \in {}^s(M)$, then $F^*(\varphi \otimes \psi) = (F^*\varphi)(F^*\psi)$, tensor field on N.

Proof:
Since two tensor fields are equal if and only if they are equal at each point, it is only necessary to see that these equations hold at each point, which follows at once from the definitions and the preceding theorem.

Corollary 3.1.10
Each $\varphi \in U^r$ including the restriction to U of any covariant tensor field on M, has a unique expression form $\varphi = \sum_{i_1} ....\sum_{i_r} a_{i_1....i_r}(\omega^{i_1} \otimes .... \otimes \omega^{i_r})$. Where at each point U, $a_{i_1....i_r} = \varphi(E_{i_1},..., E_{i_r})$ are the Components of φ in the basis $\{\omega^{i_1} \otimes ..... \otimes \omega^{i_r}\}$ and is C^∞ function on U.

3.2 : Tangent Space and Cotangent Space
The tangent space $T_p(M)$ is defined as the vector space spanned by the tangents at p to all curves passing through point p in the manifold M. And The cotangent space $T_p^*(M)$ of a manifold, at $p \in M$ is defined as the dual vector space to the tangent space $T_p(M)$. We take the basis vectors $E_i = \partial / \partial x^i$ for $T_p(M)$, and we write the basis vectors for $T_p^*(M)$ as the differential line elements $e^i = dx^i$. Thus the inner product is given by $\langle \partial / \partial x^i , dx^i \rangle = \delta_i^j$.

Definition 3.2.1 [Wedge Product.]
Carton's wedge product, also known as the exterior Product, as the ant symmetric tensor product of cotangent space basis elements $dx \wedge dy = \frac{1}{2}(dx \otimes dy - dy \otimes dx) = -dy \wedge dx$. Note that, by definition, $dx \wedge dx = 0$. The differential line elements dx and dy are called differential 1-forms or 1-form; thus the wedge product is a rule for construction g 2-forms out of pairs of 1-forms.

Definition 3.2.2
Let $\Lambda^p(x)$ be the set of anti-symmetric p-tensors at a point x .This is a vectors space of dimension $n! / p!(n-p)!$.
The $\Lambda^p(x)$ path together to define a bundle over M . $C^\infty(\Lambda^p)$ is the space of smooth p-forms, represented by anti-symmetric tensors $f_{ij...}(x)$, having p indices contracted with the wedge product of p differentials. The elements of $C^\infty(\Lambda^p)$ may then be written explicitly as follows:

$$
\begin{aligned}
&C^\infty(\Lambda^0) = \{ f(x)\} \text{ , dim } = 1 \\
&C^\infty(\Lambda^1) = \{ f_i(x)dx^i\} \text{ , dim } = \text{n} \\
&C^\infty(\Lambda^2) = \{ f_{ij}(\text{x})\text{dx}^i \wedge \text{dx}^j\} \text{ ,dim } = \text{n(n - 1)}/2! \\
&C^\infty(\Lambda^3) = \{ f_{ijk}(x)dx^i \wedge dx^j \wedge dx^k\} \text{ , dim } = \text{n(n - 1)(n - 2)}/3! \\
&C^\infty(\Lambda^{n-1}) = \{ f_{i_1...i_{n-1}} dx^{i_1} \wedge ... \wedge dx^{i_{n-1}}\} \text{ , dim } = n \\
&C^\infty(\Lambda^n) = \{ f_{i_1...i_n} dx^{i_1} \wedge ... \wedge dx^{i_n}\} \text{ , dim } = 1 \text{ .}
\end{aligned} \tag{7}
$$

Remark 3.2.3
Let α_p be an element of $\Lambda^p \alpha_p$, β_p an element of Λ^q. Then $\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p$. Hence odd forms ant commute and the wedge product of identical 1-forms will always vanish.

3.3 : Differentiable manifolds
A differentiable manifolds is necessary for extending the methods of differential calculus to spaces more general R^n a subset $S \subset R^3$ is regular surface if for every point $p \in S$ the a neighborhood V of P is R^3 and mapping $x : u \subset R^2 \to V \cap S$ open set $U \subset R^2$ such that :(i) x is differentiable homomorphism. (ii) the differentiable $(dx)_q : R^2 \to R^3$, the mapping x is called a aparametnzation of S at P the important consequence of differentiable of regular surface is the fact that the transition also example below if $x_\alpha : U_\alpha \to S^1$ and $x_\beta : U_\beta \to S^1$ are $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = w \neq \phi$, the mappings $x_\beta^{-1} \circ x_\alpha : x^{-1}(w) \to R^2$ and .

(8) $$\left(x_\alpha^{-1} \circ x_\beta\right) = x_\beta^{-1}(w) \rightarrow R$$

Are differentiable. A differentiable structure on a set M induces a natural topology on M it suffices to $A \subset M$ be an open set in M if and only if $x_\alpha^{-1}(A \cap x_\alpha(U_\alpha))$ is an open set in R^n for all α it is easy to verify that M and the empty set are open sets that a union of open sets is again set and that the finite intersection of open sets remains an open set. Manifold is necessary for the methods of differential calculus to spaces more general than de R^n, a differential structure on a manifolds M induces a differential structure on every open subset of M, in particular writing the entries of an $n \times k$ matrix in succession identifies the set of all matrices with $R^{n,k}$, an matrix of rank k can be viewed as a k-frame that is set of k linearly independent vectors in R^n, $n \times k$ $V_{n,k} K \le n$ is called the steels manifold ,the general linear group $GL(n)$ by the foregoing $V_{n,k}$ is differential structure on the group n of orthogonal matrices, we define the smooth maps function $f : M \rightarrow N$ where M, N are differential manifolds we will say that f is smooth if there are atlases (U_α, h_α) on M, (V_B, g_B) on N, such that the maps $g_B f h_\alpha^{-1}$ are smooth wherever they are defined f is a homeomorphism if is smooth and a smooth inverse. A differentiable structures is topological is a manifold it an open covering U_α where each set U_α is homeomorphism, via some homeomorphism h_α to an open subset of Euclidean space R^n, let M be a topological space , a chart in M consists of an open subset $U \subset M$ and a homeomorphism h of U onto an open subset of R^m, a C^r atlas on M is a collection (U_α, h_α) of charts such that the U_α cover M and h_B, h_α^{-1} the differentiable .

Definition 3.3.1 Differentiable injective

A differentiable manifold of dimension N is a set M and a family of injective mapping $x_\alpha \subset R^n \rightarrow M$ of open sets $u_\alpha \in R^n$ into M such that : (i) $u_\alpha x_\alpha(u_\alpha) = M$ (ii) for any α, β with $x_\alpha(u_\alpha) \cap x_\beta(u_\beta)$ (iii) the family (u_α, x_α) is maximal relative to conditions (i),(ii) the pair (u_α, x_α) or the mapping x_α with $p \in x_\alpha(u_\alpha)$ is called a parameterization , or system of coordinates of M, $u_\alpha x_\alpha(u_\alpha) = M$ the coordinate charts (U, φ) where U are coordinate neighborhoods or charts , and φ are coordinate homeomorphisms transitions are between different choices of coordinates are called transitions maps.

(9) $$\varphi_{i,j} : (\varphi_j \circ \varphi_i^{-1})$$

Which are anise homeomorphisms by definition, we usually write $x = \varphi(p), \varphi : U \rightarrow V \subset R^n$ collection U and $p = \varphi^{-1}(x), \varphi^{-1} : V \rightarrow U \subset M$ for coordinate charts with is $M = \cup U_i$ called an atlas for M of topological manifolds. A topological manifold M for which the transition maps $\varphi_{i,j} = (\varphi_j \circ \varphi_i)$ for all pairs φ_i, φ_j in the atlas are homeomorphisms is called a differentiable , or smooth manifold , the transition maps are mapping between open subset of R^m, homeomorphisms between open subsets of R^m are C^∞ maps whose inverses are also C^∞ maps , for two charts U_i and U_j the transitions mapping.

(10) $$\varphi_{i,j} = (\varphi_j \circ \varphi_i^{-1}) : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$$

And as such are homeomorphisms between these open of R^m, for example the differentiability $(\varphi'' \circ \varphi^{-1})$ is achieved the mapping $(\varphi'' \circ (\tilde{\varphi})^{-1})$ and $(\tilde{\varphi} \circ \varphi^{-1})$ which are homeomorphisms since $(A \approx A'')$ by assumption this establishes the equivalence $(A \approx A'')$, for example let N and M be smooth manifolds n and m respecpectively , and let $f : N \rightarrow M$ be smooth mapping in local coordinates $f' = \left(\psi \circ f \circ \varphi^{-1}\right) : \varphi(U) \rightarrow \psi(V)$,with respects charts (U, φ) and (V, ψ), the rank of f at $p \in N$ is defined as the rank of f' at $\varphi(p)$ (i.e) $rk(f)_p = rk(J f')_{\varphi(p)}$ is the Jacobean of f at p this definition is independent of the chosen chart , the commutative diagram in that.

(11) $$f'' = \left(\psi' \circ \psi^{-1}\right) \circ \tilde{f} \circ \left(\varphi' \circ \varphi^{-1}\right)^{-1}$$

Since $\left(\psi' \circ \psi^{-1}\right)$ and $\left(\varphi' \circ \varphi^{-1}\right)$ are homeomorphisms it easily follows that which show that our notion of rank is well defined $(J f'')_{x'} = J\left(\psi' \circ \psi^{-1}\right)_{y'} J f'\left(\varphi' \circ \varphi^{-1}\right)^{-1}$, if a map has constant rank for all $p \in N$ we simply write $rk(f)$, these are called constant rank mapping. The product two manifolds M_1 and M_2 be two C^k -manifolds of dimension n_1 and n_2 respectively the topological space $M_1 \times M_2$ are arbitral unions of sets of the form $U \times V$ where U is open in M_1 and V is open in M_2, can be given the structure C^k manifolds of dimension n_1, n_2 by defining charts as follows for any charts M_1 on (V_j, ψ_j) on M_2 we declare that $(U_i \times V_j\ ,\ \varphi_i \times \psi_j)$ is chart on $M_1 \times M_2$ where $\varphi_i \times \psi_j : U_i \times V_j \rightarrow R^{(n_1+n_2)}$ is defined so that.

(12) $$\varphi_i \times \psi_j(p, q) = (\varphi_i(p), \psi_j(q)) \text{ for all } (p, q) \in U_i \times V_j$$

A given a C^k n-atlas, A on M for any other chart (U, φ) we say that (U, φ) is compatible with the atlas A if every map $(\varphi_i \circ \varphi^{-1})$ and $(\varphi \circ \varphi_i^{-1})$ is C^k whenever $U \cap U_i \neq 0$ the two atlases A and $\tilde{A}$ is compatible if every chart of one is compatible with other atlas . A sub manifolds of others of R^n for instance S^2 is sub manifolds of R^3 it can be obtained as the image of map into R^3 or as the level set of function with domain R^3 we shall examine both methods below first to develop the basic concepts of the theory of Riemannian sub manifolds and then to use these concepts to derive a equantitive interpretation of curvature tensor , some basic definitions and terminology concerning sub manifolds, we define a tensor field called the second fundamental form which measures the way a sub manifold curves with the ambient manifold , for example X be a sub manifold of Y of $\pi : E \rightarrow X$ and $g : E_1 \rightarrow Y$ be two vector brindled and assume that E is compressible , let $f : E \rightarrow Y$ and $g : E_1 \rightarrow Y$ be two tubular neighborhoods of X in Y then there exists a C^{p-1}. The smooth manifold, an n-dimensional manifolds is a set that looks like R^n. It is a union of subsets each of which may be equipped with a coordinate system with coordinates running over an open subset of R^n. Here is a precise definition.

Definition 3.3.2

Let M be a metric space we now define what is meant by the statement that M is an n-dimensional C^∞ manifold. (i). A chart on M is a pair (U, φ) with U an open subset of M and φ a homeomorphism a (1-1) onto, continuous function with continuous inverse from U to an open subset of R^n, think of φ as assigning coordinates to each point of U . (ii) Two charts (U, φ) and (V, ψ) are said to be compatible if the transition functions.

(13) $$(\psi \circ \varphi^{-1}) : \varphi(U \cap V) \subset R^n \rightarrow \psi(U \cap V) \subset R^n \quad , \quad (\varphi \circ \psi^{-1}) : \psi(U \cap V) \subset R^n \rightarrow \varphi(U \cap V) \subset R^n$$

Are C^∞ that is all partial derivatives of all orders of $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ exist and are continuous. (ii) An atlas for M is a family $A = \{(U_i, \varphi_i) : i \in I\}$ of charts on M such that $\{U_i\}_{i \in I}$ is an open cover of M and such that every pair of charts in A are compatible. The index set I is completely arbitrary. It could consist of just a single index. It could consist of uncountable many indices. An atlas A is called maximal if every chart (U, φ) on M that is compatible with every chat of A .(iii) An n-dimensional manifold consists of a metric space M together with a maximal atlas A

Example 3.3.3

Let I_n be the identity map on R^n, then $\{R^n, I_n\}$ is an atlas for R^n indeed , if U is any nonempty open subset of R^n, then $\{U, I_n\}$ is an atlas for U so every open subset of R^n is naturally a C^∞ manifold.

Example 3.3.4

The n-space is a manifold of dimension n when equipped with the atlas $A_1 = \{(U_i, \varphi_i), (V_i, \psi_i), |1 \leq i \leq n+1|\}$ where for each $1 \leq i \leq n+1$.

(14) $$U_i = \{(x_1, \ldots, x_{n+1}) \in S^n, x_1 \geq 0\} \quad \varphi_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$$
$$V_i = \{(x_1, \ldots, x_{n+1}) \in S^n, x_1 \leq 0\} \quad \psi_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$$

So both φ_i and ψ_i just discard the coordinate x_i they project onto R^n viewed as the hyper plane $x_i = 0$. A another possible atlas, compatible with A_1 is $A_2 = \{(U, \varphi), (V, \psi)\}$ where the domains that $U = S^m \setminus \{0, \ldots, 0, 1\}$ and $V = S^n \setminus \{0, \ldots, 0, -1\}$ are the stereographic projection from the north and south poles, respectively, both φ and ψ have range R^n plus an additional single point at infinity

Example 3.3.5

The torus T^2 is the two dimensional surface $T^2 = \{(x, y, z) \in R^3, (\sqrt{x^2 + y^2} - 1)^2 + z^2 = 1/4\}$ in R^3 in cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = 0$ the equation of the torus is $(r-1)^2 + z^2 = 1/4$ fix any θ , say θ_0 . Recall that the set of all points in R^n that have $\theta = \theta_0$ is an open book, it is a hall-plane that starts at the z axis. The intersection of the tours with that half plane is circle of radius 1/2 centered on $r = 1, z = 0$ as φ runs form 0 to 2π , the point $r = 1 + 1/2\cos\varphi$ and $\theta = \theta_0$ runs over that circle. If we now run θ from $0\ to\ 2\pi$ the point $(x, y, z) = ((1+1/2\cos\varphi)\cos\theta, (1+1/2\sin\varphi)$ runs over the whole torus. So we may build coordinate patches for T^2 using θ and φ with ranges $(0, 2\pi)$ or $(-\pi, \pi)$ as coordinates).

3.4: [Intrinsic Ultracontractivily on Bounded Bomains Manifolds]

We first consider on R^d let D be connected bounded Lipchitz domain in $R^{d^n}(d \geq 1)$. And Δ with Laplacian with Dirichlet boundary conditions on D . It is well Know that the spectrum of Δ is discrete, $\sigma(-\Delta) = (\lambda_i) \geq 1$ with $0 \leq \lambda_1 \leq \lambda_2 \leq .....$, and each λ_i is an eigenvalue with finite multiplicity. Denote by P_t^D The dirichlet heat kernel on D, and $\phi \geq 0$ the first normalized Eigen function of $-\Delta$ and D it is also well known that D is intrinsically ultra-contractive (i.e).

(15) $$\zeta_t = e^{\lambda_i t} \sup_{x,y \in D} \left(\frac{p_i(x,y)}{\phi(x)\Psi(y)} \right) \langle \infty, t \rangle$$

Indeed, this is given true for more general domains such as holder domains of order o. The main purpose of this section is to clarify the short time behavior of Δ for Lipchitz domains. When D is a … domain.

(16) $$\zeta_t \leq 1 + C_t^{-(d+2)/2}, \ t \geq 0$$

Holds for some constant ≥ 0 this estimate was extended recently to smooth compact Riemannian manifolds (under some additional) geometrical assumptions) our aim is to study similar estimate for less smooth domains D. we shall see that the estimate, holds for $C^{1,\alpha}$ domains for any $\alpha \geq 0$, If D is merely lipchitzian (i.e) $C^{1,0}$ is no larger true. For instance, for $D = (0,1)^d$ one has $\phi(x) = \prod_{k=1}^d P_t^{(0,1)}(x_i, y_i)$, $\phi(x) = \prod_{k=1}^d \sin(\pi, x_i)$. and where $\sin(\pi r)$ is the first dirichlet Eigen function on [0,1]. Thus combining this with below for, $D = (0,1)$ we obtain.

(17) $$\left(\frac{1}{2} t^{-3d/2} \leq \zeta_t \leq C t^{-3d/2} \right), \ t \in (0,1]$$

For some constant $C \geq 0$. A natural question is therefore whether for Lipchitz domain there exists a constant $C \geq 0$ such that:

(18) $$\zeta_t \leq 1 + Ct \ , \ t \geq 0$$

We shall see that the answer is no, in general .It is true that $\zeta_t \leq 1 + Ct^{-p}$. For some (qualitative) constant of the boundary.

We prove that for any $B \geq 0$, there exists alipschitz (connected) domain D such that $t^B \zeta_t$ is not bounded $t \to 0$ we summarize this as well as the large time behavior and a lower that domain D is called Lipschitzian if for any $x \in \partial D$. There exist $S \geq 0$ a coordinate system is called (Lipschilzian), $(r, \theta) \in R \times R^{d-1}$, and a Lipchitz function f on R^{d-1} such that x is the origin and.

(19) $$\begin{Bmatrix} B(x,s) \cap D = B(x,s) \cap \{(r,\theta) : r \geq f(\theta)\} \\ B(x,s) \cap \partial D = B(x,s) \cap \{(r,s) : r = f(\theta)\} \end{Bmatrix}$$

A Lipchitz domain is called $C^{1,\alpha}$ for some $\alpha \geq 0$, if the corresponding Lipchitz function satisfies.

(20) $$\left| \nabla f(a) - V(b) \right| \leq C \left| a - b \right|^\alpha$$

for some $C \geq 0$ and for all, $a, b \in R^{d-1}$. In this definition it is required that ≥ 2 , if $d = 1$, D is an open bounded interval.

Theorem 3.4.1

If D is a $C^{1,\alpha}$ - domain for some ≥ 0 , there exists a constant $C \geq 0$, such that.

(21) $$\max\left\{1, \frac{1}{C} t\right\}^{-(a+2)/2} \leq \zeta_t \leq 1 + C(\wedge t) e^{-(d+2)}, \ for\ all\ t \geq 0$$

For any $B \geq 0$, there exists a bounded Lipchitz domain $D \subset R$ such that : $\lim_{t \to 0} Sup\ t^B \zeta_t = +\infty$. Now let M be a d-dimensional connected Riemannian manifolds and D an open bounded $C^{1,1}$ domain in M. then for any $x \in \partial D$

there exist $S \geq 0$, a local coordinate system in $(r,\theta) \in R \times R^{d-1}$ in $B(x,s)$. (The open geodesic ball at x with radius, s) and $f \in C_b^1(R^{d-1})$ with bounded second derivatives such that holds. For any.

(22) $$y = (r,\theta) \in B(x,s) \cap \overline{D} \quad \text{Define} \quad f(y) = r - f(\theta) \geq 0$$

Then $y = (r,\theta) \in B(x,s) \cap \overline{D}$ has bounded second order derivatives furthermore there exists a constant $C \geq 0$ such that.

(23) $$p(y) \leq C\left|(r,\theta) - f(\theta,\theta)\right| = cF\ (y)$$

Where ρ is the Riemann $\rho \geq \rho_1$, $on\ D$ Nina distance to ρ . This by the partition of unity, there exists a non-negative function $\overline{\rho} \in C_b^1(\overline{D})$ with bounded derivative and $\rho\ \big|_{\partial D} = 0$ such that:

(24) $$\rho \geq \rho_1 \quad , \quad on \quad D$$

For some constant $1 \geq 0$, since $\overline{D}$ is compact for simplicity we may and do assume that M is compact to.

(25) $$L = N \sum_{i=1} X_i^2 + X$$

Where X is a bounded measurable vector field and $\{X_i\}_{i=1}^N$ are C^1 vector fields we assume that L is elliptic that is.

(26) $$\begin{Bmatrix} (f,f) = \sum_i^N (X_i, f)^2 \geq \left|\nabla f\right|^2 \quad , \quad f \in C^1 \\ \mu(f^2) \leq r\,\mu(\left|\nabla f\right|^2) + B(r)\,\mu(\psi)\left|f\right|^2, r \geq 0,\ f,\ f \in C^1(M) \end{Bmatrix}$$

For some constant $2 \geq 0$. Thus under a local coordinate system on has.

(27) $$L = \sum_{i=1}^d a_{ij} \partial_i \partial_j + \sum_{i=1}^d b_i \partial_i$$

Where $(a_{ij})_{d \times d}$ is continuous and strictly positive definite $b_i(1 \leq i \leq d)$ are bounded measurable. The L-diffusion process uniquely exists. For any $\in D$, Let $(X_t(x))$ be the L-diffusion process starting from x and, $T(x) = \inf\{t \geq 0, X_t(x) \in \partial D\}$ .For all bounded measurable function f on D .To study the (intrinsic ultracontractivity) of PD_t . We assume that L is symmetric w.r.t a probability measure $\mu(dx)$, $\mu(dx) = 1, D^{v(x)}dx$ where V is abounded. Measurable function and (dx) the Riemannian volume measure by the elasticity and the sobolev in equality, we know that spectrum of L on D with dirichlet boundary condition is discrete, As in section 1, we let $\lambda_1 \geq \lambda_2$ be the first two dirichlet eigenvalues and $\phi \geq 0$ the normalized first Eigen function.

Remark 3.4.2 [Exterior Derivative]

The exterior derivative operation, which takes p -forms into $(p+1)$ -forms according to the rule :

(28) $$C^\infty(\Lambda^0) \xrightarrow{d} C^\infty(\Lambda^1) \ ;\ d(f(x)) = \frac{\partial f}{\partial x^i} dx^i \ ,\ C^\infty(\Lambda^1) \xrightarrow{d} C^\infty(\Lambda^2) \ ;\ d(f_j(x)dx^j) = \frac{\partial f_j}{\partial x^i} dx^i \wedge dx^j$$

$$C^\infty(\Lambda^2) \xrightarrow{d} C^\infty(\Lambda^3) \ ;\ d(f_{jk}(x)dx^j \wedge dx^k) = \frac{\partial f_j}{\partial x^i} dx^i \wedge dx^j \wedge dx^k$$

Here we have taken the convention that the new differential line element is always inserted before any previously existing wedge products.

Property 3.4.3

An important property of exterior derivative is that it gives zero when applied twice: $d\ dw_p = 0$. This identity follows from the equality of mixed partial derivative, as we can see from the following simple example:

(29) $$C^\infty(\Lambda^0) \xrightarrow{d} C^\infty(\Lambda^1) \xrightarrow{d} C^\infty(\Lambda^2)$$

$$df = \partial_j f dx^j,\ ddf = \partial_i \partial_j f dx^i \wedge dx^j = \frac{1}{2}(\partial_i \partial_j f - \partial_j \partial_i f) dx^i \wedge dx^j = 0.$$

Remark 3.4.4

(i) The rule for differentiating the wedge product of a p -form α_p and a q -form β_q is $d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q$. (ii) The exterior derivative anti-commutes with 1-forms.

Examples 3.4.5

Possible p -forms α_p in two-dimensional space are:

(30)
$$\begin{Bmatrix} \alpha_0 = f(x,y) \\ \alpha_1 = u(x,y)dx + v(x,y)dy \\ \alpha_2 = \phi(x,y)dx \wedge dy. \end{Bmatrix}$$

The exterior derivative of line element gives the two-Dimensional curl times the area $d(u(x,y)dx + v(x,y)dy) = (\partial_x v - \partial_y u)dx \wedge dy$.

The three-space p -forms α_p are.

(31)
$$\begin{Bmatrix} \alpha_0 = f(x) \\ \alpha_1 = v_1 dx^1 + v_2 dx^2 + v_3 dx^3 \\ \alpha_2 = w_1 dx^2 \wedge dx^3 + w_2 dx^3 \wedge dx^1 + w_3 dx^1 \wedge dx^2 \\ \alpha_3 = \phi(x) dx^1 \wedge dx^2 \wedge dx^3. \end{Bmatrix}$$

We see that

(32)
$$\begin{Bmatrix} \alpha_1 \wedge \alpha_2 = (v_1 w_1 + v_2 w_2 + v_3 w_3) dx^1 \wedge dx^2 \wedge dx^3 \\ d\alpha_1 = (\varepsilon_{ijk} \partial_j v_k) \frac{1}{2} \varepsilon_{ijm} dx^1 \wedge dx^m \\ d\alpha_2 = (\partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3) dx^1 \wedge dx^2 \wedge dx^3. \end{Bmatrix}$$

(Where ε_{ijk} is the totally anti-symmetric tensor in 3-dimensions).

Definition 3.4.6

An alternating covariant tensor field of order r on M will be called an exterior differential form of degree r (or some time simply, r -form).The set $\Lambda^r(M)$ of all such forms is a subspace of $(M)^r$.

Theorem 3.4.7

Let $\wedge(M)$ denote the vector space over R of all exterior differential forms. Then for $\varphi \in \Lambda^r(M)$ and $\psi \in \Lambda^s(M)$, the formula, $(\varphi \wedge \psi)_P = \varphi_P \wedge \psi_P$ defines an associative product satisfying $\varphi \wedge \psi = (-1)^{rs} \psi \wedge \varphi$. With this product, $\wedge(M)$ is algebra over R . If $f \in C^\infty(M)$, we also have $f(\varphi\psi) = \varphi \wedge (f\psi) = \varphi \wedge (f\psi)$ If $\omega^1,..., \omega^n$ is a field of co frames on M (or an open set U of M), then the set

(33)
$$\left\{ \left(\omega^{i_1} \wedge ... \wedge \omega^{i_r}\right) \;/\; \left(1 \le i_1 < i_2 < ... < i_r \le n\right) \right\}$$

is a basis of $\Lambda^r(M)$ or $\wedge(U)$.

Theorem 3.4.8

If $F: M \to N$ is a C^∞ mapping of manifolds, then $F^*: \wedge(N) \to \wedge(M)$ is an algebra homomorphism. (We shall call $\wedge(M)$ the algebra of differential forms or exterior algebra on M).

Definition 3.4.8

An oriented vector space is a vector space plus an equivalence class of allowable bases, choose a basis to determine the orientation those equivalents to it will be called oriented or positively oriented bases or frames. This concept is related to the choice of a basis Ω of $\Lambda^n(V)$.

Lemma 3.4.9

Let $\Omega \ne 0$ be an alternating covariant tensor on V of order, n = dim V and let $e_1,..., e_n$ .be a basis of V . Then for any set of vectors $v_1,..., v_n$, with $v_i = \Sigma \alpha_i^j e_j$, , we have , $\Omega(v_1,...v_n) = \det \; (\alpha_i^j) \;\; \Omega \;\; (e_1,..., e_n)$.

Proof:

This lemma says that up to a non-vanishing scalar multiple Ω is the determinant of the components of its variables. In particular, if $V = V^n$ is the space on n-tuples and $e_1,..., e_n$ is the canonical basis, then $\Omega(v_1,..., v_n)$ is proportional to the determinant whose rows are $v_1,..., v_n$. The proof is a consequence of the definition of determinant. Given Ω and $v_1,..., v_n$, we use the linearity and ant symmetry of Ω to write.

(34)
$$\Omega \; (v_1,..., v_n) = \Sigma \, \alpha_1^{j_1} ... \alpha_n^{j_n} \, \Omega \;\; \left(e_{j_1},..., e_{j_n}\right).$$

Since $\Omega\left(e_{j_1},..., e_{j_n}\right)=0$, if two indices are equal, we may write. $\Omega\ (v_1,..., v_n)=\sum \operatorname{sgn}\ \sigma\ (\alpha_1^{\sigma(1)}...\alpha_n^{\sigma(n)})\Omega\ \ (e_1,..., e_n)=\det\ (\alpha_i^j)\ \ \Omega\ \ (e_1,..., e_n)$ .The last equality uses the standard definition of determinant.

Corollary 3.4.10

Note that if $\Omega \neq 0$, then $v_1,..., v_n$ are linearity independent if and only if $\Omega\ (v_1,..., v_n)\neq 0$. Also note that the formula of the lemma can be construed as a formula for change of component of Ω , there is just one component since $\Lambda^n(V)=1$, when we change from the basis $(e_1,..., e_n)$ of V to the basis $(v_1,..., v_n)$ .These statements are immediate consequences of the formula in the lemma.

Definition 3.4.11

We shall say that M is orient able it is possible to define a $C^{\infty}\ n-$ form Ω on M which is not zero at any point, in which case M is said to be oriented by the choice of Ω . A manifold M is orient able if and only if it has a covering $\{U_\alpha, \varphi_\alpha\}$ of coherently oriented coordinate neighborhoods.

Theorem 3.4.12

Let M be any C^{∞} Manifold and let $\Lambda(M)$ be the algebra of exterior differential forms on M .Then there exists a unique R -linear map $d_M:\Lambda(M)\rightarrow\Lambda(M)$ such that.

(i) If $f\in\Lambda^{\circ}(M)=C^{\infty}(M)$, then $d_M\ f=df$, the differential of f .

$$\theta\in\Lambda^r(M)\ \ and\ \ \sigma\in\Lambda^s(M),\ then\ \ d_M\left(\theta\wedge\sigma\right)=d_M\theta\wedge\sigma+(-1)^r\theta\wedge d_M\sigma$$

(ii) $d_M^2=0$ .This map will commute with restriction to open sets $U\subset M$, that is, $(d_M\theta)_U=d_U\theta_U$, and map $\Lambda^r(M)$ into $\Lambda^{r+1}(M)$.

IV. Conclusion

The paper study Riemannian differenterentiable manifolds is a generalization of locally Euclidean E^n in every point has a neighbored is called a chart homeomorphism, so that many concepts from as differentiability manifolds. We give the basic definitions, theorems and properties of Laplacian Riemannian manifolds becomes the spectrum of compact support M and Direct commutation of the spectrum, and spectral geometry of operators de Rahm.

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