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Two Ways of Stability Analysis of Prey-Predator System with Diseased Prey Population

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ABSTRACT

In this paper, the dynamics of a discrete-time prey-predator system with a disease in the prey population is analyzed. The existence, the boundedness and the stability of equilibrium points were studied algebraically and proved that the system is qualitatively stable. The purpose of this work is to provide a mathematical framework to study the response of a predator – preymodel to a disease in the prey population and harvesting of diseased prey. Numerical Simulations reveal that the reasonable harvesting prevents the spread of disease.

Keywords: *Discrete Model, Qualitative Stability, Diseased Prey, Equilibrium Points, Harvesting*

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I. Introduction

Many examples of a predator – prey interaction among species can be easily observed in ecological system throughout the world. In normal life, predator and prey species exhibit regular cycles of abundance or population increase and decrease. The dynamics of predator – prey interactions have been studied extensively in the recent years by researchers [1-3]. Epidemiological models have received much more attention from many researchers and these models care about the analytical study of the spread of infectious diseases between the species. The key role in epidemiological models is played by the 'incidence rate' which is a function describing the mechanism of transmission of the disease from infectious individual to susceptible individual. Several epidemic models with such different types of incidence rates have been extensively studied by many researchers [4-6]. This paper will explore the dynamics of such epidemiological system. Although the number is still limited, some modified predator – prey models with disease have been introduced, for example, the disease in prey, predators consume only infected preys, predators avoid infected prey, the disease in predators only, predators consume both healthy and infected preys [7-11,13, 19].Also, this paper investigates the complex effects in discrete time prey-predator model with harvesting on diseased prey.

II. The Model

We make the following assumptions to formulate the mathematical model of prey-predator system with diseased prey population.

(i) The predator consumes the susceptible prey only.
(ii) The infected prey is harvested.
(iii) There is no other food supply for the predator other than the susceptible prey.
(iv) The predator cannot be infected and if there is no susceptible prey, then the predator will die.
(v) The prey population grows according to logistic equation.
(vi) Theinfected prey populationdoes not recover from the disease.

Based on the assumptions, the proposed mathematical model is formulated as follows.

$$\begin{aligned}\frac{dx_1}{dt} &= ax_1\left(1-\frac{x_1}{k_1}\right)- \alpha x_1x_2 - p_1x_1y\\ \frac{dx_2}{dt} &= bx_2\left(1-\frac{x_2}{K_2}\right)+ \alpha x_1x_2 - hx_2 \qquad (1)\\ \frac{dy}{dt} &= cx_1y - dy\end{aligned}$$

With the initial densities $x_1(0) > 0$, $x_2(0) > 0$ and y(0) > 0. Herex_1(t), x_2(t) and y(t) denote the numbers of Susceptible prey, Infected prey and Predator respectively and parameters are all positive.
Model parameters are described below.

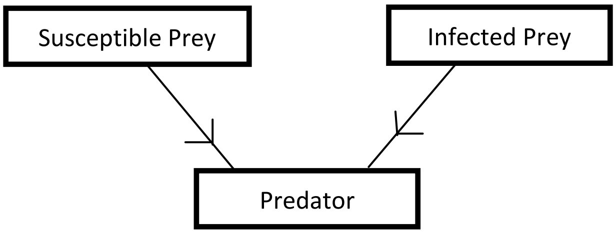
Parameters	Description
a	Logistic growth of susceptible prey (S.Prey)
b	Logistic growth of infected prey (I.Prey)
K_1	Carrying capacity of susceptible prey
K_2	Carrying capacity of infected prey
P_1	Predation rate of susceptible prey
α	Rate of infection of disease
h	Rate of harvesting of infected prey
c	Growth rate of predator due to its interaction with the susceptible prey
d	Natural death rate of predator

III.Analysis

We investigate the stabilityofsystem (1) using both qualitative and quantitative stability conditions.

3.1 Qualitative Analysis

This section deals with the stability of system (1) considering only the structure of the food web. The food web for this system is as follows:



Model structure is described by an iteration matrix which summarizes the interactions between species in the web and for n species, this matrix has n^2 elements. Each pairwise element in this matrix, a_{ij} , can be negative, zero, or positive (that is, a_{ij} -, 0,or +) depending on whether the population of species i is decreased, is unaffected, or is increased by the presence of species j [18, 13].

The iteration matrix for system (1) is given by

$$P = \begin{bmatrix} - & - & - \\ + & - & 0 \\ + & 0 & - \end{bmatrix}$$

Weinvestigatethestabilityof system (1) using the following conditions.

Qualitative Stability Conditions: Mathematically, the necessary and sufficient conditions for the existence of qualitative stability [13] in an n×n matrix, A are

i) $a_{ii} \leq 0$ for all i.
ii) $a_{ii} \neq 0$ for atleast one i.
iii) the product $a_{ij}a_{ji} \leq 0$ for all $i \neq j$.
iv) For any sequence of three or more indices, i, j, k, …..,q, r (with $i \neq j \neq k \neq \ldots\ldots \neq q \neq r$), the product $a_{ij} . a_{jk} \ldots\ldots a_{qr} . a_{ri} = 0$.
v) The determinant of the matrix, $\det A \neq 0$.

If the conditions (i) – (v) are not satisfied, then it does not imply that the matrix is unstable, but rather that a complete knowledge about the magnitude of the interaction coefficients is needed.
It is clear that the matrix P satisfies all the required conditions and hence the system (1) is qualitatively stable.

3.2 Quantitative Analysis

Many people have made numerical analysis of stability conditions of specific multispecies communities[13,20]. These works have considered the actual magnitudes of the interactions between species in the system.In this section, we analyze the complex behavior of the model (1) using linear stability conditions.

3.2.1 Boundedness of the System

In this section, we discuss the solution of the system(1) when it is bounded.

Theorem: 1 The Sound prey population is bounded.
Proof:

From (1), we have

$$\frac{dx_1}{dt} = ax_1\left(1 - \frac{x_1}{K_1}\right) - \alpha x_1 x_2 - p_1 x_1 y$$
$$< \frac{ax_1}{K_1}\ (K_1 - x_1)$$

By simple argument, we have $\lim_{n\to\infty} sup\ x_1(t) < K_1$. Hence, the sound prey population is bounded.

3.2.2 Equilibrium Points

In this section, we will study the stability behavior of the system (1) at equilibrium points which are given below.

i) Trivial Equilibrium: E_0 (0, 0, 0)
ii) Axial Equilibrium: E_1 (K_1, 0, 0)
iii) Boundary Equilibrium: $E_2\left(\frac{d}{c}, \frac{a\,(cK_1-d\,)}{\alpha cK_1}, 0\right)$
iv) Boundary Equilibrium: $E_3\left(\frac{h-b}{\alpha}, 0, \frac{a\,(\,\alpha K_1-h+b\,)}{K_1 p_1\,(\,h-b\,)}\right)$
v) Boundary Equilibrium: $E_4\left(0, K_2\left(\frac{b-h}{b}\right), \frac{ab-\alpha K_2+\alpha K_2 h}{bp_1}\right)$
vi) Positive Interior Equilibrium: $E_5\left(\frac{d}{c}, \frac{K_2}{b}\left(b + \frac{\alpha d}{c} - h\right), \frac{a}{p_1}\left(1 - \frac{d}{cK_1}\right) - \frac{\alpha K_2}{p_1 b}\left(b + \frac{\alpha d}{c} - h\right)\right)$

3.2.3 Linear stability Analysis

At this stage, we analyze the local behavior of the model (1) around each fixed point.
The Jacobian matrix of the model (1) at state variable is given by

$$J = \begin{bmatrix} a - \frac{2ax_1}{K_1} - \alpha x_2 - p_1 y & \alpha x_2 & cy \\ -\alpha x_1 & b - \frac{2bx_2}{K_2} + \alpha x_1 - h & 0 \\ -p_1 x_1 & 0 & cx_1 - d \end{bmatrix}$$

Now, we use the linearized stability technique for quantitative analysis of the non-linear system (1).

Theorem: 2

Let $p(\lambda) = \lambda^3 + B\lambda^2 + C\lambda + D$. There are atmost three roots of the equation $p(\lambda) = 0$. Then the following statements are true:

a) If every root of the equation has absolute value less than one, then the fixed point of the system is locally asymptotically stable and fixed point is called a sink.
b) If at-least one of the roots of the equation has absolute value greater than one, then the fixed point of the system is unstable and fixed point is called saddle.
c) If every root of the equation has absolute value greater than one, then the system is source.
d) The fixed point of the system is called hyperbolic if no root of the equation has absolute value equal to one. Ifthere exists a root of the equation with absolute value equal to one, then the fixed point is called non-hyperbolic.

3.2.4 Complex behavior of the Model

In this part, we investigate the local stability around each fixed point of the system (1).

Proposition: 1 The trivial equilibrium point E_0(0, 0, 0) is locally asymptotically stable only when a < 1 and b < h.
Proof: The Jacobian matrix J (E_0) at E_0 is

$$J(E_0) = \begin{bmatrix} a & 0 & 0 \\ 0 & b-h & 0 \\ 0 & 0 & -d \end{bmatrix}$$

The eigen values of J (E_0) at E_0 are λ_1 = a, λ_2 = b – h, and λ_3 = -d. By theorem (1), it follows that E_0 is locally asymptotically stable if a < 1 and b < h.

Proposition: 2 The axial equilibrium point E_1(K_1, 0, 0) is locally asymptotically stable if c $<\frac{1+d}{K_1}$ and

$b < 1 + h - \alpha K_1$.

Proof: The Jacobian matrix J (E_1) at E_1 is given by

$$J(E_1) = \begin{bmatrix} -a & 0 & 0 \\ -\alpha K_1 & b + \alpha K_1 - h & 0 \\ -p_1 K_1 & 0 & cK_1 - d \end{bmatrix}$$

One of the eigen values of the above matrix is $\lambda_1 = cK_1 - d$. Remainingeigen values are obtained from the following characteristic Equation

$$\lambda^2 + (a - b - \alpha K_1 + h)\lambda - a(b + \alpha K_1 - h) = 0$$

andtheireigen values are $\lambda_2 = b + \alpha K_1 - h$ and $\lambda_3 = -a$.

E_1 is locally asymptotically stable if $|\lambda_{1,2,3}| < 1$.This holds when $cK_1 - d < 1$ and $b + \alpha K_1 - h < 1$. Therefore, E_1 is locally asymptotically stable if and only if $c < \frac{1+d}{K_1}$ and $b < 1 + h - \alpha K_1$.

Dynamic behavior of the Model around the fixed point E_2: In this section, we analyze the stability of the system(1) at $E_2(\frac{d}{c}, \frac{a(cK_1 - d)}{\alpha c K_1}, 0)$. The Jacobian matrix J (E_2) at E_2 is given by

$$J(E_2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where$a_{11} = \frac{-ad}{cK_1}$, $a_{12} = a - \frac{ad}{cK_1}$, $a_{13} = 0$, $a_{21} = \frac{-\alpha d}{c}$, $a_{22} = b - \frac{2ab}{\alpha K_2} + \frac{2abd}{c\alpha K_1 K_2} + \frac{\alpha d}{c} - h$, $a_{23} = 0$, $a_{31} = \frac{-dp_1}{c}$, $a_{32} = 0$ and $a_{33} = 0$.

The Characteristic Equation of J (E_2) is $\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$, where $A_1 = -(a_{11} + a_{22} + a_{33})$, $A_2 = a_{11}a_{12} - a_{12}a_{21} - a_{23}a_{32} + a_{11}a_{33}$, and $A_3 = a_{11}a_{23}a_{32} - a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33}$.

According to the Routh–Hurwitz criterion, $E_2(\frac{d}{c}, \frac{a(cK_1 - d)}{\alpha c K_1}, 0)$ is locally asymptotically stable only when $A_1 > 0$, $A_3 > 0$ and $A_1 A_2 > A_3$[14, 15].

Proposition: 3 Theboundary equilibrium $E_3(\frac{h-b}{\alpha}, 0, \frac{a(\alpha K_1 - h + b)}{K_1 p_1 (h-b)})$ is locally asymptotically stable only when $c(h - b) < \alpha(1 + d)$ and $\frac{\alpha K_1}{h-b} + \frac{2(h-b)}{\alpha} > K_1(1 - \frac{1}{a})$.

Proof: The Jacobian matrix J(E_3) at E_3 is given by

$$J(E_3) = \begin{bmatrix} \beta & 0 & \frac{ca(\alpha K_1 - h + b)}{K_1 p_1 (h-b)} \\ b - h & 0 & 0 \\ \frac{p_1(b-h)}{\alpha} & 0 & c(\frac{h-b}{\alpha}) - d \end{bmatrix}$$

The eigen values of the above matrix are $\lambda_1 = c(\frac{h-b}{\alpha}) - d$, $\lambda_2 = 0$ and $\lambda_3 = \beta$, where

$\beta = \frac{\alpha a K_1 h - \alpha a b K_1 - 2a(h-b)^2 - \alpha(a\alpha K_1 - ah + ab)}{K_1 \alpha (h-b)}$. By theorem (1), the fixed point E_3 is stable only if $|\lambda_{1,3}| < 1$. This is possible only when $c(\frac{h-b}{\alpha}) - d < 1$ and $\beta < 1$.

Now,$\beta < 1$ implies that $\frac{\alpha a K_1 h - \alpha a b K_1 - 2a(h-b)^2 - \alpha(a\alpha K_1 - ah + ab)}{K_1 \alpha (h-b)} < 1$.

$$\Rightarrow \quad \alpha K_1 h - \alpha b K_1 - 2(h-b)^2 - \alpha(\alpha K_1 - h + b) < \frac{K_1 \alpha (h-b)}{a}$$

$$\Rightarrow K_1(h-b) - \frac{2(h-b)^2}{\alpha} + (h-b) < \frac{K_1(h-b)}{a} + \alpha K_1$$

$$\Rightarrow \frac{\alpha K_1}{h-b} + \frac{2(h-b)}{\alpha} > K_1 - \frac{K_1}{a}$$

$$\Rightarrow \frac{\alpha K_1}{h-b} + \frac{2(h-b)}{\alpha} > K_1(1 - \frac{1}{a})$$

Therefore, the boundary equilibrium $E_3(\frac{h-b}{\alpha}, 0, \frac{a(\alpha K_1 - h + b)}{K_1 p_1 (h-b)})$ is locally asymptotically stable only when $c(h - b) < \alpha(1 + d)$ and $\frac{\alpha K_1}{h-b} + \frac{2(h-b)}{\alpha} > K_1(1 - \frac{1}{a})$.

Proposition: 4 The boundaryequilibrium $E_4(0, K_2(\frac{b-h}{b}), \frac{ab - \alpha K_2 b + \alpha K_2 h}{bp_1})$ is locally asymptotically stable only if $h < b$.

Proof: The Jacobian matrix J(E_4) at E_4 is given by

$$J(E_4) = \begin{bmatrix} 0 & \alpha K_2\left(\frac{b-h}{b}\right) & \frac{abc-\alpha K_2 bc+\alpha K_2 hc}{bp_1} \\ 0 & h-b & 0 \\ 0 & 0 & -d \end{bmatrix}$$

The eigen values of the above matrix are $\lambda_1 = 0$, $\lambda_2 = h - b$ and $\lambda_3 = -d$.
$E_4$is locally asymptotically stable if $|\lambda_{1,2,3}|< 1$.This is possible only when $h - b < 1$. Hence, E_4 is locally asymptotically stable when $h < b$.

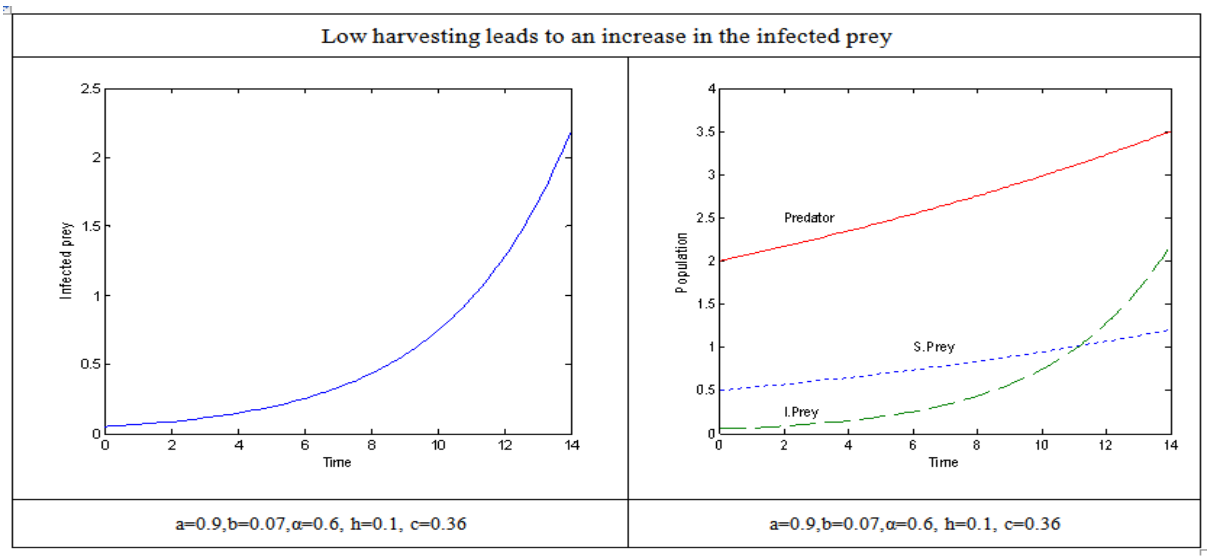
Dynamic behavior of the Model around the fixed point E_5: In this section, we analyze the stability of the system(1) at $E_5\left(\frac{d}{c}, \frac{K_2}{b}\left(b+\frac{\alpha d}{c}-h\right), \frac{a}{p_1}\left(1-\frac{d}{cK_1}\right)-\frac{\alpha K_2}{p_1 b}\left(b+\frac{\alpha d}{c}-h\right)\right)$
The Jacobian matrix $J(E_5)$ at E_5 is given by

$$J(E_2) = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

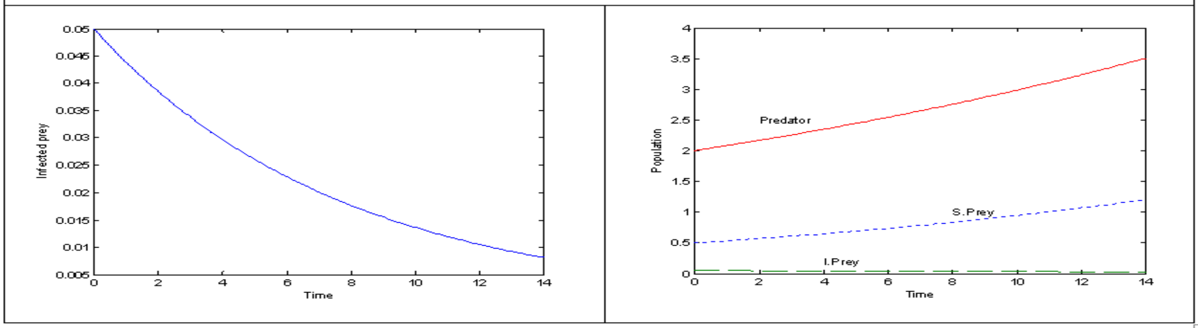
Where $b_{11}=\frac{-\alpha d}{cK_1}$, $b_{12} = \frac{\alpha K_2}{b}\left(b+\frac{\alpha d}{c}-h\right)$, $b_{13} = \frac{ca}{p_1}\left(1-\frac{d}{cK_1}\right)-\frac{\alpha c K_2}{p_1 b}\left(b+\frac{\alpha d}{c}-h\right)$, $b_{21} = \frac{-\alpha d}{c}$, $b_{22} = -b - \frac{\alpha d}{c} + h$,
$b_{23} = 0$, $b_{31}=\frac{-p_1 d}{c}$, $b_{32} = 0$, and $b_{33} = 0$.
The eigen values of the above matrix are $\lambda_1=\frac{-\alpha d}{cK_1}$, $\lambda_2 = -b -\frac{\alpha d}{c} + h$ and $\lambda_3 = 0$.
E_4 is locally asymptotically stable if $|\lambda_{1,2,3}|< 1$.This is possible only when $-b - \frac{\alpha d}{c} + h < 1$.Therefore, the system(1) shows local asymptotic stability at E_5 only when $h < b +\frac{\alpha d}{c}$.

IV. Numerical Simulation

In this section, all our important analytical findings are numerically verified using MATLAB. First of all, we fixed all parameters to ensure all populations survive. Then, we find the effect of harvest on the disease. Numerical simulations explain the effect of the parameters on the behavior of the three classes of populations and also guarantees that all solutions of the system lie within the region. It is also observed that the infected prey increases when there is a low harvest and decreases when there is a large harvest.



High harvesting leads to a decrease in the infected prey



V. Conclusion

In this paper, we have developed an epidemiological prey - predator model where only the prey population is infected by an infectious disease. In order to maintain a healthy population, the infected prey was harvested. We have shown that the proposed system is qualitatively stable. Conditions for stability at various equilibrium points were obtained. It is also observed that the increase in harvest affects the disease and thus prevents the occurrence of an epidemic. We have performed numerical simulation for the positive equilibrium of the proposed system. So, all important analytical findings are numerically verified by using MATLAB. Finally, we conclude that the system (1) of prey - predator model with infectious disease in the prey population exhibits very interesting dynamics. Venturino considered recovery from the disease, which is not considered herein [16]. So, as a part of future work to improve the model, we can incorporate this factor in the proposed model to make it more realistic.

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