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μ-RANGE, λ-RANGE OF OPERATORS ON A HILBERT SPACE

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ABSTRACT

We have introduced a new type of operator called "Trijection Operator" on a linear space. It is a generalization of projection. We study trijection in case of a Hilbert space. Further we decompose range of a trijection into two disjoint sub-spaces called μ-range and λ- range and study their properties.

Keywords: Operator, Projection, Trijection Operator, Hilbert space, μ-range and λ- range.

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I. INTRODUCTION

Trijection Operator: We define an operator E to be a trijection if $E^3 = E$. Clearly if E is a projection, then it is also a trijection for

$$E^2 = E \Rightarrow E^3 = E^2.E = E.E = E^2 = E.$$

Thus it follows that a projection is necessarily a trijection. But a trijection is not necessarily a projection. This would be clear from the complex of trijections given below.

Example of Trijections:

Consider R^2. Let z be an element in R^2. Thus z = (x , y) where x , y $\in$ R.

Let E(z) = (ax + by, cx + dy), where a, b, c, d are scalars.

By calculation we find that –

$$E^3(z) = (a_1x + b_1y, c_1x + d_1y)$$

Where,

$$a_1 = a^3 + 2abc + bcd$$
$$b_1 = a^2b + abd + b^2c + bd^2$$
$$c_1 = a^2c + acd + bc^2 + cd^2$$

and $d_1 = abc + 2bcd + d^3$.

Hence if E is a trijection, $E^3 = E$, then we have

$$a = a^3 + 2abc + bcd$$
$$b = a^2b + abd + b^2c + bd^2$$
$$c = a^2c + acd + bc^2 + cd^2$$

and $d = abc + 2bcd + d^3$.

Case(i) :- For some particular cases, let a = d = 0, then $b = b^2c$, $c = bc^2$.

Thus we can choose b = c =1 and get

$$E(z) = (y, x)$$
$$E(x,y) = (y, x) \qquad \ldots\ldots\ldots\ldots\ldots..(1)$$

So, $E^2(x , y) = E(E(x , y)) = E(x , y) = (x , y)$ [from(1)]

Therefore $E^2(x , y) = (x , y) \neq (y , x) = E(x , y)$

Hence $E^2 \neq E$

Thus E is not a projection.

But $E^3(x , y) = E(E^2(x , y)) = E(x , y)$

Therefore $E^3 = E$

Hence E is a trijection.

Case(ii):- If a = c =0, then $b = bd^2$, $d=d^3$.

Thus we can choose d = 1 and get E(z) = (by , y).

Therefore E(x , y) = (by , y)

Hence $E^2(x, y) = E(E(x, y)) = E(by, y) = E(x, y)$.
Thus $E^2 = E$.
Hence E is a projection, and so it is also a trijection.

Case(iii):- If $a = b = 0$ then $c = cd^2$, $d = d^3$.
Thus choosing d = 1, we have $E(z) = (0, cx + y)$
$E(x, y) = (0,cx + y)$.
Therefore $E^2(x ,y) = E(E(x, y)) = E(0, cx + y)$
$= (0, cx + y) = E(x, y)$.
Thus $E^2 = E$.
Hence E is a projection and so it is a trijection.

Case(iv):- If $b = c = 0$ then $a = a^3$, $d = d^3$. Now $a = a^3$ gives a = 0,1 or -1. Similarly we get values of d. We consider these values in a systematic way.

$a = d = 0 \Rightarrow E = 0$, it is the zero mapping. Hence it is a projection and trijection also.
$a = 0, d = 1 \Rightarrow E(z) = (0, y)$. It is also a projection.
$a = 0, d = -1 \Rightarrow E(z) = (0, -y)$. It is also a projection.
$a = 1, d = 0 \Rightarrow E(z) = (x, 0)$. It is also a projection.
$a = 1, d = 1 \Rightarrow E = (I)$, the identity mapping which is also a projection and a trijection.
$a = 1, d = -1 \Rightarrow E(z) = (x, -y)$, which is not a projection.
$a = -1, d = 0 \Rightarrow E(z) = (-x, 0)$, which is not a projection.
$a = -1, d = 1 \Rightarrow E(z) = (-x, y)$, which is not a projection.
$a = -1, d = -1 \Rightarrow E(z) = (-x, -y)$, which is not a projection.

In this way, a projection is necessarily a trijection but a trijection is not a projection.

Trijection in a Hilbert Space:- We first define a normed linear space:

A normed linear space is a linear space N in which to each vector x there corresponds a real number, denoted by $\| x \|$ and called the norm of x, in such a manner that

1. $\| x \| \geq 0$ and $\| x \| = 0 \Leftrightarrow x = 0$.
2. $\| x + y \| \leq \| x \| + \| y \|$, for $x, y \in N$.
3. $\| \alpha x \| = \| \alpha \| . \| x \|$, for scalar.

N is also a metric space with respect to the metric d defined by $d(x,y) = \| x - y \|$. A Banach space is a complete normed linear space.[1]

A trijection on a Banach space b is defined as an operator E on B such that $E^3 = E$ and E is continuous.

μ-Range and λ-Range of a Trijection: If E is a trijection on linear space H, then we know that $H = R \oplus N$ where R is the range of E and N is the null space of E. We can decompose R into two subspaces L and M such that $L = \{z : E(z) = z\}$ and $M = \{z : E(z) = -z\}$ and $L \cap M = \{0\}$.

Now as L , M are parts of the range R, let us call L as λ-Range and M as μ-Range of E. Thus the range R of E is the direct sum of its λ and μ-Ranges. In case E is a projection, we clearly see that its range coincides with its λ-Range while μ-Range is {0}.

II. THEOREMS

Theorem 1: If E is a trijection on a Banach space B and if R and N are its range and null space, then R and N are closed linear subspaces of B such that

$$B = R \oplus N.$$

Proof: Since the null space of any continuous linear transformation is closed, so N being null space of E is closed.

Since $E^2 - I$ is also continuous and
$R = \{z : E^2 z = z\}$
$= \{z : (E^2 - I) z = 0\}$
So R is also closed. Clearly,
$B = R \oplus N$.

A Hilbert space is a complex Banach space whose norm arises from an inner product, that is, in which there is defined a complex function (x , y) of vectors x and y with the following properties:-

1. $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$
2. $(\bar{x}, \bar{y}) = (y, x)$
3. $(x, x) = \| x \|^2$.

Where x , y , z are elements of Banach space and α is a scalar.[2]

To each operator T on a Hilbert space H there corresponds a unique mapping T* of H into itself (called the adjoint of T*) which satisfies the relation

$$(T x, y) = (x, T^*y)$$

For all x , y in H. Following are the properties of adjoint operations:-

1. $(T_1 + T_2)^* = T_1^* + T_2^*$
2. $(\alpha T)^* = \bar{\alpha}\ T^*$
3. $(T_1\ T_2)^* = T_2^* + T_1^*$
4. $T^{**} = T$
5. $\|T^*\| = \|T\|$
6. $\|T^*T\| = \|T\|^2$.[3]

Theorem2: If E is a trijection on a Hilbert space H then so is E*.

Proof: Since E is a trijection on a Hilbert space H, so $E^* = E$ and $E^3 = E$.

If T , U , V are linear operators on H, then

$$(T U V)^* = (T (U V))^* = (U V)^*T^* = V^* U^* T^*$$

Hence letting T = U = V= E, we have

$$(E^3)^* = E^*.E^*E^* = (E^*)^3$$

Therefore $E^* = (E^*)^3$.

Now $(E^*)^* = E^{**} = E = E^*$, so E* satisfies the conditions for trijection on a Hilbert space. Hence E* is a trijection.

Theorem3: If A is an operator on a Hilbert space H, then

$$(A^k)^* = (A^*)^k \qquad \text{…………. (1)}$$

For K being 0 or any positive integer. Moreover, if A is self-adjoint then so is A^k.

Proof: The equation (1) is obviously true when k = 0 or 1. Now we prove the equation (1) by the method of induction.

Let us assume that the equation (1) is true for K – 1. Therefore,

$$(A^{k-1})^* = (A^*)^{k-1}$$

Therefore,
$$\begin{aligned}(A^k)^* &= (A^{k-1}.A)^* \\ &= A^*.(A^{k-1})^* \\ &= (A^*).(A^*)^{k-1} \\ &= (A^*)^k.\end{aligned}$$

Hence the proof is complete by induction.

Theorem4: If E is a trijection on a Hilbert space H, then $I - E^2$ is also trijection on H such that

$$R_{I-E^2} = N_E \text{ and } N_{I-E^2} = R_E.$$

Proof: Since
$$\begin{aligned}(I - E^2)^2 &= (I - E^2)(I - E^2) \\ &= I - E^2 - E^2 + E^4 \\ &= I - E^2 - E^2 + E^3.E \\ &= I - E^2 - E^2 + E^2 \\ &= I - E^2,\end{aligned}$$

Therefore
$$\begin{aligned}(I - E^2)^3 &= (I - E^2)^2 (I - E^2) \\ &= (I - E^2)(I - E^2) \\ &= (I - E^2)^2 \\ &= I - E^2\end{aligned}$$

Also
$$\begin{aligned}(I - E^2)^* &= I^* - (E^2)^* \\ &= I - (E^*)^2 \\ &= I - E^2.\end{aligned}$$

Hence $I - E^2$ is a projection as well as a trijection on H.

Moreover,

$$\begin{aligned}R_{I-E^2} &= \{z : (I - E^2)^2 z = z\} \\ &= \{z : (I - E^2) z = z\} \\ &= \{z : z - E^2 z = z\} \\ &= \{z : E^2 z = 0\} \\ &= N_E, \text{ and}\end{aligned}$$

$$\begin{aligned}N_{I-E^2} &= \{z : (I - E^2) z = 0\} \\ &= \{z : z - E^2 z = 0\} \\ &= \{z : E^2 z = z\} \\ &= R_E.\end{aligned}$$

Theorem5: Any trijection on a Hilbert space H can be expressed as the sum of two self-adjoint operators on H such that one of them is a projection and the square of the other is the identity operator.

Proof: Let E be a trijection on H. Then we can write

$$E = (I - E^2) + E^2 + E - I$$

By theorem (4), $I - E^2$ is a projection on H, and $I - E^2$ is a self-adjoint.

Also $(E^2 + E - I)^* = E^2 + E - I$, so $E^2 + E - I$ is self-adjoint. Moreover we have

$$\begin{aligned}(E^2 + E - I)^2 &= (E^2 + E - I)(E^2 + E - I)\\ &= E^4 + E^3 - E^2 + E^3 + E^2 - E - E^2 - E + I\\ &= E^2 + E - E^2 + E + E^2 - E - E^2 - E + I\\ &= I, \text{ Identity operator.}\end{aligned}$$

This proves the theorem.

Two vectors x and y in a Hilbert space H are said to be orthogonal (written $x \perp y$) if $(x,y) = 0$. A vector x is said to be orthogonal to a non-empty set S (written $x \perp S$) is $x \perp y$ for every y in S, and the orthogonal complement of S, denoted by $S^{\perp}$, is the set of all vector orthogonal to S.[4]

An operator N and H is said to be normal if it commutes with its adjoint, that is, if

$$NN^* = N^*N.$$ [5]

Theorem6: If E is a trijection on a Hilbert space H, then

$$x \in R_E \Leftrightarrow E^2x = x \Leftrightarrow \|E^2x\| = \|x\|.$$

Also $\|E\| \leq 1$.

Proof: From theorem(5), the first equivalence is clear.

Also $E^2x = x \Rightarrow \| E^2x \| = \|x\|$.

So we now need to show that

$\| E^2x \| = \|x\| \Rightarrow E^2x = x$.

Now $\|x\|^2 = \| E^2x + (I - E^2)x\|^2$

$= \|y + z\|^2$, say where $y = E^2x$ and $z = (I - E^2)x$. Also,

$$\begin{aligned}\|y + z\|^2 &= (y + z, y + z)\\ &= (y , y + z) + (z, y + z)\\ &= (y , y) + (y , z) + (z , y) + (z , z)\\ &= \|y\|^2 + \|z\|^2 + (y , z) + (z , y).\end{aligned}$$

Since $y = E^2x = E(Ex) \in R_E$ and

$z = (I - E^2)x \in N_E = (R_E)^{\perp}$, hence (y,z) = (z,y) = 0.

Therefore $\|y + z\|^2 = \|y\|^2 + \|z\|^2$ ....(1)

$$\begin{aligned}\text{Now } \| E^2x \| = \|x\| &\Rightarrow \| E^2x \|^2 = \|x\|^2\\ &\Rightarrow \| (I - E^2)x\|^2 = 0 \quad \text{[from (1)]}\\ &\Rightarrow \| (I - E^2)x\| = 0\\ &\Rightarrow (I - E^2)x = 0\\ &\Rightarrow x = E^2x.\end{aligned}$$

Also from(1), for any x in H, we have

$$\begin{aligned}\| E^2x \|^2 \leq \|x\|^2 &\Rightarrow \| E^2x \|^2 \leq \|x\|\\ &\Rightarrow \| E^2 \| \leq 1.\end{aligned}$$

Since E is also normal operator,

$\| E^2 \| = \| E \|^2$.[6]

Therefore $\| E^2 \| \leq 1$.

Hence $\| E \| \leq 1$.

Theorem7: If E is a trijection on a Hilbert space H then E^2 is projection on H with the same range and null space as that of E.

Proof: Since $(E^2)^2 = E^4 = E^3.E = E.E = E^2$ and by theorem(3), E^2 is self-adjoint, E^2 is projection on H.

Also $R_{E^2} = \{z : E^2z = z\} = R_E$.[7]

$N_{E^2} = \{z : E^2z = 0\} = \{z : Ez = 0\} = N_E$.[8]

"Let T be an operator on E. A closed linear subspace M of H is said to be invariant under T if $T(M) \subseteq M$. If both M and $M^{\perp}$ are invariant under T, we say that M reduces T or that T is reduced by M."[9]

According to reference (9) the theorem(8) as follows:

Theorem8: If P and Q are the trijection on closed linear subspaces M and N of H, then PQ is a trijection $\Leftrightarrow$ PQ = QP. Also in this case PQ is the trijection on $M \cap N$ and $N_{PQ} = N_P + N_Q$.

Proof: Let PQ = QP, then

$(PQ)^2 = (PQ)(QP) = P(QP)Q = P(PQ)Q = (P.P)(Q.Q) = P^2Q^2$

So $(PQ)^3 = (PQ)^2 (PQ) = P^2Q^2 PQ = P^2Q(QP)Q = P^2Q(PQ)Q$

$= P^2(QP)Q^2 = P^2(PQ)Q^2 = (P^2P)(Q^2Q) = P^3Q^3.$

$= PQ$ [Since P and Q are trijection]

Also $(PQ)^* = Q^*P^* = QP = PQ$

Hence PQ is trijection.

Conversely let PQ be a trijection, then

$(PQ)^* = PQ \Rightarrow Q^*P^* = PQ.$

$\Rightarrow QP = PQ.$

Now again is PQ is a trijection, then

$R_{PQ} = \{ x : (PQ)^2x = x\} = \{x : (P^2Q^2)x = x\}$

Let $x \in M \cap N$, then $x \in M$ and $x \in N$

$\Rightarrow P^2x = x$ and $Q^2x = x$.

Hence $(PQ)^2 x = (P^2Q^2)x = P^2 (Q^2)x = P^2 x = x$

$\Rightarrow x \in R_{PQ}$.

Therefore $M \cap N \subseteq R_{PQ}$. …….(1)

Again let $x \in R_{PQ}$, then $(P^2Q^2)x = x$.

Therefore $x = P^2 (Q^2)x = P(P Q^2 x) \in M$.

Since $(QP)^2 = Q^2P^2$, $(PQ)^2 = P^2Q^2$ and QP =PQ, we have

$Q^2P^2 = P^2Q^2$.

Hence $x = P^2Q^2 x = Q^2P^2 x = Q(QP^2x) \in N$.

Now $x \in M$ and $x \in N$, so $x \in M \cap N$.

Therefore $R_{PQ} \subseteq M \cap N$ …….(2)

From (1) and (2), we get

$R_{PQ} = M \cap N = R_P \cap R_Q$.

Also $N_{PQ} = \{z : (PQ)^2z = 0 \} = \{z : (P^2Q^2)z = 0\}$

Let $z \in N_{PQ}$, then $P^2 (Q^2 z) = 0$.

Therefore $Q^2z \in N_P{}^2 = N_P$ as P is a trijection.

Since $Q(z - Q^2z) = Qz - Q^3z = Qz - Qz = 0$, so $z - Q^2z \in N_Q$.

Therefore $z = Q^2z + (z - Q^2z) \in N_P + N_Q$.

Thus $z \in N_{PQ} \Rightarrow z \in N_P + N_Q$.

Hence $N_{PQ} \subseteq N_P + N_Q$ ……..(3)

Let $z \in N_P + N_Q$, then we can write

$z = z_1 + z_2$ where $z_1 \in N_P$ and $z_2 \in N_Q$.

$\Rightarrow Pz_1 = 0$ and $Qz_2 = 0$.

Now $(PQ)z = (PQ)(z_1 + z_2) = PQz_1 + PQz_2 = Qpz_1 + PQz_2$

$= Q(Pz_1) + P(Qz_2) = Q(0) + P(0) = 0.$

Therefore $z \in N_{PQ}$.

Thus $z \in N_P + N_Q \Rightarrow z \in N_{PQ}$.

Hence $N_P + N_Q \subseteq N_{PQ}$ ..….(4)

From (3) and (4), we get

$N_{PQ} = N_P + N_Q$.

Theorem9: If P and Q are trijection on a Hilbert space H and PQ = 0, then P + Q is also trijection such that the null space pf P + Q is the intersection of the null spaces of P and Q, and the range of P + Q is the direct sum of the ranges of P and Q.

Proof: We have

$(P + Q)^2 = (P + Q)(P + Q) = P^2 + PQ + PQ + Q^2$

Now $PQ = 0 \Rightarrow (PQ)^* = 0^* \Rightarrow Q^*P^* = 0 \Rightarrow QP = 0$

Therefore PQ = QP = 0.

Hence $(P + Q)^2 = P^2 + Q^2$.

Now $(P + Q)^3 = (P + Q)(P + Q)^2$

$= (P + Q)(P^2 + Q^2)$

$= P^3 + PQ^2 + QP^2 + Q^3$
$= P^3 + (PQ)Q + (QP)P + Q^3$
$= P + Q.$

And $(P + Q)^* = P^* + Q^*$
$= P + Q.$

Hence P + Q is a trijection.

Now we are to prove that,

$N_{P+Q} = N_P \cap N_Q$

And $R_{P+Q} = N_P \oplus N_Q$

We have

$N_{P+Q} = \{z : (P + Q)z = 0\}$
$= \{z : Pz + Qz = 0\}$
$= \{z : Pz = -Qz\},$

Let $z \in N_{P+Q}$, Since PQ = 0,

$0 = (PQ)z = P(Qz)$
$= P(-Pz)$
$= -P^2z.$

Therefore $P^2z = 0$. Hnece $z \in N_P$

Again since QP = 0, therefore,

$0 = (QP)z = Q(Pz)$
$= Q(-Qz)$
$= -Q^2z.$

Thus $Q^2z = 0$. Hence $z \in N_Q$.

Therefore $z \in N_P \cap N_Q$ ………………(i)

Let $z \in N_P \cap N_Q$, then $z \in N_P$ and $z \in N_Q$

$\Rightarrow Pz = 0 = Qz$
$\Rightarrow Pz = 0 = -Qz$
$\Rightarrow (P + Q)z = 0$
$\Rightarrow z \in N_{P+Q}.$

Thus $z \in N_P \cap N_Q \Rightarrow z \in N_{P+Q}$.

Therefore $N_P \cap N_Q \subseteq N_{P+Q}$. ………………..(ii)

Hence from (i) and (ii), we get

$N_{P+Q} = N_P \cap N_Q$.

Let z be an element in R_{P+Q}, then

$(P + Q)^2z = z$
$\Rightarrow (P^2 + Q^2)z = z.$

Now as P and Q are trijections on H, so

$P^2z = P(Pz) \in R_P$ and
$Q^2z = Q(Qz) \in R_Q.$

Hence $P^2z + Q^2z \in R_P + R_Q.$

$\Rightarrow z \subseteq R_P + R_Q.$

Hence $R_{P+Q} \subseteq R_P + R_Q$. ……………….(iii)

Conversely, let $z \in R_P + R_Q$ then we can write

$z = z_1 + z_2$ such that $z_1 \in R_P$ and $z_2 \in R_Q$.

Hence $P^2z_1 = z_1$ and $Q^2z_2 = z_2$.

Therefore $(P + Q)^2z = (P^2 + Q^2)(z_1 + z_2)$
$= P^2z_1 + P^2z_2 + Q^2z_1 + Q^2z_2$
$= P^2z_1 + P^2(Q^2z_2) + Q^2(P^2z_1) + Q^2z_2$
$= z_1 + (PQ)^2z_2 + (QP)^2z_1 + z_2$
$= z_1 + z_2 = z$ [as PQ = 0]

Hence $z \in R_{P+Q}$.

Thus $z \in R_P + R_Q \Rightarrow z \in R_{P+Q}$.

Therefore $R_P + R_Q \subseteq R_{P+Q}$. ……….(iv)

From(iii) and (iv) we get,

$R_{P+Q} = R_P + R_Q$.

Now let $z \in R_P \cap R_Q$, then $P^2z = z$, $Q^2z = z$.

Therefore $z = P^2z = P^2(Q^2z)$

$= (P^2Q^2)z$

$= (PQ)^2z$

$= 0$

Hence $R_P \cap R_Q = \{0\}$.

Therefore $R_{P+Q} = R_P \oplus R_Q$.

Thus we see that with given conditions, PQ is also a trijection.

Theorem10: If E is a trijection on a Hilbert space H, then $½(E^2 + E)$ and $½(E^2 – E)$ are also trijections whose null spaces are L_6 and L_5 respectively.

Proof: We have,

$$[½(E^2 + E)]^2 = ½(E^2 + E) . ½(E^2 + E)$$
$$= ¼ (E^4 + E^2 + 2E^3)$$
$$= ¼ (E^3.E + E^2 + 2E^3)$$
$$= ¼ (E.E + E^2 +2E)$$
$$= ¼ (2E^2 + 2E)$$
$$= ½(E^2 + E),$$

and,

$$[½(E^2 - E)]^2 = ½(E^2 - E) . ½(E^2 - E)$$
$$= ¼ (E^4 + E^2 - 2E^3)$$
$$= ¼ (E^3.E + E^2 - 2E^3)$$
$$= ¼ (E.E + E^2 - 2E)$$
$$= ¼ (2E^2 - 2E)$$
$$= ½(E^2 - E),$$

also

$$[½(E^2 \pm E)]^* = ½[(E^2)^* \pm E^*]$$
$$= ½(E^2 \pm E).$$

Thus $½(E^2 + E)$ and $½(E^2 – E)$ are projections on H.

Hence they are also trijections on H. Moreover,

$$N_{½(E2 + E)} = \{z : ½(E^2 + E) z = 0\}$$
$$= \{z : (E^2 + E) z = 0 \}$$
$$= \{ z : E^2z = -Ez\}$$
$$= L_6.$$

and $$N_{½ (E2 – E)} = \{ z : ½ (E^2 – E)z = 0\}$$
$$= \{ z : (E^2 - E) z = 0 \}$$
$$= \{ z : E^2z = Ez = 0 \}$$
$$= L_5.$$

Theorem11: If E_1,E_2 are commuting trijections on a linear space H, then the λ-range of E_1 coincides with the μ-range of E_2, and vice-versa, if and only if

$$E_1 = -E_1^2E_2 \text{ and } E_2 = -E_1E_2^2.$$

Proof: Let the λ-range of E_1 be L_1 and μ-range of M_1 . Similarly we denote λ and μ - ranges of E_2 by L_2 and M_3 respectively.

Let $L_1 \subseteq M_2$.

Let z be an element in H, then since $E_1z + E_1^2z$ is in L_1, it is also in M_2.

Hence $E_2 (E_1z + E_1^2z) = -(E_1z + E_1^2z)$

$\Rightarrow E_2 E_1z + E_2 E_1^2z + E_1z + E_1^2z = 0$

$\Rightarrow E_2 E_1 + E_2 E_1^2 + E_1 + E_1^2 = 0$

Now if $E_2 E_1 + E_2 E_1^2 + E_1 + E_1^2 = 0$ and z be in L_1, then

$$E_1z = z \text{ and } E_1^2z = z.$$

Since $E_2 E_1z + E_2 E_1^2z + E_1z + E_1^2z = 0$, we have

$E_2 (E_1z) + E_2 (E_1^2z) + E_1z + E_1^2z = 0$

$\Rightarrow$ $E_2z + E_2z + z + z = 0$

$\Rightarrow$ $2E_2z + 2z = 0$

$\Rightarrow$ $E_2z = -z$

$\Rightarrow$ $z \in M_2$.

Thus $z \in L_1 \Rightarrow z \in M_2$.

Hence $L_1 \subseteq M_2$.

Therefore $L_1 \subseteq M_2 \Leftrightarrow E_2 E_1 + E_2 E_1^2 + E_1 + E_1^2 = 0$

Let $M_2 \subseteq L_1$, Now for any z in H,

$E_2z - E_2^2z \in M_2$.
$\Rightarrow$ $E_2z - E_2^2z \in L_1$
$\Rightarrow$ $E_1(E_2z - E_2^2z) = E_2z - E_2^2z$
$\Rightarrow$ $E_1E_2z - E_1E_2^2z = E_2z - E_2^2z$
$\Rightarrow$ $E_1E_2 - E_1E_2^2 = E_2 - E_2^2$.
Now if $E_1E_2 - E_1E_2^2 = E_2 - E_2^2$ and $z \in M_2$, then
$E_2z = -z$ and $E_2^2z = z$.
Since $E_1E_2z - E_1E_2^2z = E_2z - E_2^2z$
$E_1(E_2z) - E_1(E_2^2z) = E_2z - E_2^2z$
$\Rightarrow$ $E_1(-z) - E_1z = -z - z \Rightarrow -2E_1z = -2z$
$\Rightarrow$ $z \in L_1$.
Thus $z \in M_2 \Rightarrow z \in L_1$.
Hence $M_2 \subseteq L_1$.
Therefore $M_2 \subseteq L_1 \Leftrightarrow E_1E_2 - E_1E_2^2 - E_2 + E_2^2 = 0$.
Since E_1, E_2 commute and $L_1 = M_2$, we have
$E_1E_2 + E_1^2E_2 + E_1 + E_1^2 = 0$
and $E_1E_2 - E_1E_2^2 - E_2 + E_2^2 = 0$.
Therefore $L_1 = M_2 \Leftrightarrow -E_1E_2 = E_1 + E_1^2 + E_1^2E_2$
$= E_2^2 - E_2 - E_1E_2^2$ ……… (i)
Let $M_1 \subseteq L_2$. For any z in H.
$E_1z - E_1^2z \in M_1 \Rightarrow E_1z - E_1^2z \in L_2$
$\Rightarrow E_2(E_1z - E_1^2z) = E_1z - E_1^2z$
$\Rightarrow E_2E_1z - E_2E_1^2z = E_1z - E_1^2z$
$\Rightarrow E_2E_1 - E_2E_1^2 = E_1 - E_1^2$.
Now if $E_2E_1 - E_2E_1^2 = E_1 - E_1^2$ and z is in M_1, then
$E_1z = -z$ and $E_1^2z = z$
Since $E_2E_1z - E_2E_1^2z = E_1z - E_1^2z$, we have
$E_2(E_1z) - E_2(E_1^2z) = E_1z - E_1^2z$
$\Rightarrow E_2(-z) - E_2z = -z - z$
$\Rightarrow -2E_2z = -2z$
$\Rightarrow E_2z = z$
$\Rightarrow z \in L_2$.
Thus $z \in M_1 \Rightarrow z \in L_2$.
Hence $M_1 \subseteq L_2$.
Therefore $M_1 \subseteq L_2 \Leftrightarrow E_2E_1 - E_2E_1^2 = E_1 - E_1^2$.
Similarly $L_2 \subseteq M_1 \Leftrightarrow E_1E_2 + E_1E_2^2 + E_2 + E_2^2 = 0$.
Since E_1, E_2 commute and $L_2 = M_1$, we have
$L_2 = M_1 \Leftrightarrow -E_1E_2 = E_1^2 - E_1 + E_1^2E_2$
$= E_2 + E_2^2 + E_1E_2^2$ ……………. (ii)
Since $L_1 = M_2$ and $L_2 = M_1$,
From (i) and (ii), we get
$E_1^2 - E_1 + E_1^2E_2 = E_1 + E_1^2 + E_1^2E_2$ and
$E_2 + E_2^2 + E_1E_2^2 = E_2^2 - E_2 - E_1E_2^2$.
Hence $E_1 + E_1^2E_2 = 0$ and $E_2 + E_1E_2^2 = 0$
Therefore $E_1 = -E_1^2E_2$ and $E_2 = -E_1E_2^2$.
Conversely suppose $E_1 = -E_1^2E_2$ and $E_2 = -E_1E_2^2$, then
$E_1^2 = (-E_1^2E_2)^2 = E_1^4E_2^2$
$= E_1(E_1E_2^2) = E_1(-E_2) = -E_1E_2$.
Hence $E_2^2 = (-E_1E_2)^2 = E_1^2E_2^4 = E_1^2E_2^2$
$= -E_1E_2$.
Therefore $E_1 + E_1^2 + E_1^2E_2 = (E_1 + E_1^2E_2) + E_1^2$
$= E_1^2$
$= -E_1E_2$.
and $E_2^2 - E_2 - E_1E_2^2 = E_2^2 - (E_2 + E_1E_2^2)$
$= E_2^2$

$= - E_1 E_2.$

Hence from (i), we have

$$L_1 = M_2.$$

Again as,

$$E_1^2 - E_1 - E_1^2 E_2 = E_1^2 - (E_1 + E_1^2 E_2)$$
$$= E_1^2$$
$$= - E_1 E_2,$$

and $E_2 + E_2^2 + E_1 E_2^2 = (E_2 + E_1E_2^2) + E_2^2$
$$= E_2^2$$
$$= - E_1 E_2.$$

Hence from (ii), we get,

$$L_2 = M_1.$$

Thus we have proved,

$$L_1 = M_2 \text{ and } L_2 = M_1.$$

Thus the λ – range of E_1 coincides with μ-range of E_2 and vice-versa has been proved.

III. CONCLUSION

In this way, I have introduced a new type operator, called "Trijection Operator" on a linear space. It is a generalization of projection operator. I have studied and examined trijection in case of Hilbert space. Further I have decomposed range of a trijection into two disjoint subspaces called μ-range, λ–range and studied their properties.

An operator E on a linear space L is called a trijection if $E^3 = E$. It is a generalization of projection operator in the sense that every projection is a trijection but a trijection is not necessarily a projection. Then a trijection on a Hilbert space is an operator which is satisfying the condition and is also self-adjoint. Some theorems concerning trijection operator on a Hilbert space have been proved.

If P and Q are the trijection on a closed liner a subspaces M and N of H, then

$$PQ \text{ is a trijection } \Leftrightarrow PQ = QP,$$

Also in this case PQ is the trijection on $M \cap N$ and $N_{PQ} = N_P + N_Q$.
If E is a trijection on H, then ½(E^2 + E) and ½(E^2 – E) are also trijection whose all null spaces are: $L_6 = \{z : E^2z = - Ez\}$ and $L_5 = \{z : E^2z = Ez\}$ respectively.

Next I have presented range of a trijection into two disjoint sub-spaces called μ-range, λ–range. These ranges are defined as follows: If E is a trijection on a linear space H, then I can decompose range R into two sub-spaces L and M such that $L = \{z : E(z) = z\}$ and $M = \{ z:E(z) = -z\}$ and $L \cap M = \{0\}$. I say L as λ–range and M as μ-range of E. If E_1, E_2 are commuting trijection on a linear space H, then the λ–range of E_1 coincides with the μ-range of E_2 and vice-versa, if and only if, $E_1 = -E_1^2E_2$ and $E_2 = -E_1E_2^2$.

Thus a new operator, trijection operator on a linear space has been studied with Hilbert space and two disjoint subspaces μ-range and λ–range.

REFERENCES

[1]. Simmons. G.F. : Introduction to topology and Modern Analysis (1973,2008),P.212.

[2]. Ibid, P.245.

[3]. Ibid, P.265.

[4]. Ibid, P.249.

[5]. Ibid, P.269.

[6]. Ibid, P.271.

[7]. Theorem (4), of this paper.

[8]. Ibid (Theorem 4), of this paper.

[9]. Simmons. G.F. : Introduction to topology and Modern Analysis (1973,2008),P.275.

[10]. Dunford, Nelson and Schwartz Jacob, T.: Linear Operators Part I: General theory,[John Wiley and Sons, New York],1988.

[11]. Rudin, W.: Functional Analysis, [McGraw-Hill Book company, Inc., NewYork],1973, 2004.

[12]. Kreyszig, Erwin.: Introductory Functional Analysis with Application,[John Wiley and Sons, New York],1989, Pp.469-471.

[13]. Jha, K.K. : Functional Analysis,[Students' friends, patna],1986.

[14]. Taylor, A.E. : Introduction To Functional Analysis,[Wiley, New York],1958.2004.

[15]. Yoshida, K. : Functional Analysis [Springer-Verlag, New York Inc., New York], 1968,2000.