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k-Separable non-inclusion and m-Non-inclusion Techniques for $r(r \leq n)$ Enumerative arrangements

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ABSTRACT

Let $X = \{x_t : t = 1, 2, \ldots, N\}$ be a finite collection and let $K = \{x_{ti} : i = 1, 2, \ldots, k\}$ and $M = \{x_{tj} : j = 1, 2, \ldots, m\}$ be sub-collections of X. Furthermore, we considered the problem of selecting $r(r \leq n)$ elements from X with the k and m non-inclusion elements, such that the k non-inclusion elements are not always together in each selection. We constructed suitable mathematical formula in the combinatorial sense for enumerative purpose on certain kind of restriction called k-separable non-inclusion and m-non-inclusion on the sub-collections of elements of the set $X = \{x_t : t = 1, 2, \ldots, N\}$.

KEYWORDS: R-permutations, r-combinations, k-inseparable non-inclusion and m-non-inclusion. Mathematics Subject Classi_cation: 05A10

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I. INTRODUCTION AND PRELIMINARIES

Let $X = \{x_i : i = 1, 2, \ldots, N\}$ be a finite collection and let $K = \{x_{ij} : j = 1, 2, \ldots, k\}$ be a sub-collection of X.Consider the problem of selecting $r(r \leq n)$ elements from X in such a way that each selection;

(i) Contains the entire k-elements of the sub-collection K $(r \geq k)$; we call this the inclusion case.

(ii) Contains only some part of K and not the entire k-elements; we call this the non-inclusion case.

(iii) Contains the entire k-elements but in such a way that the k-elements are not together (separate); we call this the k-separable inclusion.

(iv) Contains the entire k-elements but in such a way that the k-elements are always together; we call this the k-inseparable inclusion.

(v) Contains only some part of K and not the entire k-elements but in such a way that the k-elements are always not together (separate); we call this the k-separable non-inclusion.

(vi) Contains only some part of K and not the entire k-elements but in such a way that the k-elements are always together; we call this the k-inseparable non-inclusion.

We consider various arrangements of the elements in X, such that, a fixed group k of elements is given different restrictions. If N is small (say 2, 3 or 4) it is easy to exhaustively list and count all the possible outcomes in this arrangements for either the inclusion case or non-inclusion case.

Furthermore, for any given collections of r-arrangements, observe that for sufficiently large value of N there are various sub-collections of arrangements which are of interest but without any well-known mathematical formula in literature. Some of these sub-classes of r-arrangements are as described in i, to vi above, however an efficient mathematical formula have been provided for the problems in i, to vi above (see[42, 43, 44, 45, 46, 47, 48, 49]).In fact, it is quit obvious that as N increases we can define many more of these sub-collections of r-arrangements.

So far, to the best of my knowledge most of the standard Text in combinatorics has failed to address this concept,even when it is introduced it is left with a vacuum that relegate this concept to the background of no important (see[1, 2, 3, 4, 6, 12, 13]). For this reason, In this research work we shall consider more than one sub-collections of X with a prescribe restrictions on the elements of these sub-collections of X with the aim of providing enumerative formulas associated with the sub-collections of r-arrangements.

Let $X = \{x_i : i = 1, 2, \dots, N\}$ be a finite collection and let $K = \{x_j : j = 1, 2, \dots, k\}$ and $M = \{x_l : l = 1, 2, \dots, m\}$ be a distinct sub-collections of X. Consider the problem of selecting $r(r \leq n)$ elements from X in such a way that each selection contains some part of k-elements with each of this element next to each other and also contains only some part of m-elements (not the entire m-elements) ; we call this the k-Separable-non-inclusion and m-non-inclusion

Lemma 1.1(principle of inclusion and exclusion) (see e.g [2,3,5]) If $(A_1,\ A_2,\ \cdots,\ A_k)$ is any sequence of finite sets, then

$$n\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{\substack{I\subset[k]\\ I\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{i\in I} A_i\right)$$

Results

Theorem 2.1

Let $X = \{x_t : t = 1, 2, ..., N\}$,then the r-permutation of N distinct elements $(r \leq N)$ with the k-separable noninclusion and m-noninclusion all at a time is

$$P_{(n,r,k,m)} = \begin{cases} \displaystyle\sum_{i=\alpha_1(N,r,k,m)}^{\beta_1(N,r,k,m)} \sum_{j=\alpha_2(N,r,k,m)}^{\beta_2(N,r,k,m)} i!j!(r-i-j)! \binom{N-k-m}{r-i-j} \binom{k}{i} \binom{m}{j} \\ \displaystyle\binom{r-i}{j} \binom{r-i+1}{i}, \quad i,\ j \neq 0 \\ \\ r! \displaystyle\binom{N-k-m}{r}, \quad i,\ j = 0 \end{cases}$$

Where the values of $\alpha_\ell(N, r, k, m)$ and $\beta_l(N, r, k, m)$ $\{\ell = 1,\ 2\}$ depend on which of the condition(s) 1 to 10 given below are satisfied.

(1) $k \leq r,\ m \leq r,\ m + k \leq r \ and \ r + k + m \leq N$
(2) $k \leq r,\ m \leq r,\ m + k \leq r \ and \ r + k + m > N$
(3) $k \leq r,\ m \leq r,\ m + k > r \ and \ r + k + m > N$
(4) $k \leq r,\ m \leq r,\ m + k > r \ and \ r + k + m \leq N$
(5) $k \leq r,\ m > r \ and \ r + k + m > N$
(6) $k \leq r,\ m > r \ and \ r + k + m \leq N$
(7) $k > r,\ m \leq r \ and \ r + k + m > N$
(8) $k > r,\ m \leq r \ and \ r + k + m \leq N$
(9) $k > r,\ m > r \ and \ r + k + m > N$
(10) $k > r,\ m > r \ and \ r + k + m \leq N$

Proof

To prove this theorem we shall suppose that

(1)$K \neq \emptyset \ and \ M \neq \emptyset$, now we partition the set X such that

$X = (X \backslash (K \cup M)) \cup K \cup M$

Let n(M)=m, n(K)=k, so that $n(X \backslash (K \cup M)) = N - k - m$

Now we shall begin by considering the followings;

(2) $\mathbf{k \leq r,\ m \leq r}$ and $\mathbf{m + k \leq r}$.

In this case certainly, there is an r-permutation that will include the entire fixed k-elements of the set K and are next to each other, that is these elements are always together (inseparable). However, we ensure that this r-permutation does not include the entire m-elements of the set M as well. . To ensure this, we consider the following arrangements;

(a): Suppose $\mathbf{N-k-m \geq r}$.

Observe that the first and last elements of the set K can be arrange in k and $k-i+1$ ways while the first and last elements of the set M could be arranged in m and $m-j+1$ ways, which gives $P_{(k,i)}$, $P_{(m,j)}$ respectively. Consequently the first and last elements of the set $(X\backslash(K\cup M))$ will be arrange in $(N-k-m)$ and $(N-r-m-k+i+j+1)$ ways, which gives $P_{(N-k-m\,,\,r-i-j)}$

Observe that all the arrangements we have done are yet to be in r-ordering, which is the desired ordering or length. However, we required that this r-ordered elements will satisfy the prescribed conditions on the sets $K\ and\ M$. To achieve this, we consider further the following arrangements which will bring together the previous arrangements we have done.

Thus for any jth-ordered arrangement in $P_{(m,j)}$ to fit (rearranged) with any fixed $(r-i-j)th$-ordered arrangement in $P_{(N-k-m\,,\,r-i-j)}$ we shall have their first and last fitting ways to be $(r-i-j+1)$ and $r-i$ ways, which gives $\frac{(r-i)!}{(r-i-j)!}$. Finally, we fit in the ith-ordered arrangements in $P_{(k,i)}$ in the resultant $(r-i)th$-ordered rearrangement of $P_{(m,j)}$ and $P_{(N-k-m\,,\,r-i-j)}$ with their first and last fitting ways to be $(r-i+1)$ and $(r-2(i-1))$ ways, which gives $\frac{(r-i+1)!}{(r-2i+1)!}$ which is the desired r-ordering of elements of X satisfying the prescribed conditions. At this juncture, we shall apply First Counting Principle (FCP) to the various arrangements we have done, so that we shall have altogether

$$P_{(k,i)}P_{(m,j)}P_{(N-k-m,r-i-j)}\frac{(r-i)!}{(r-i-j)!}\frac{(r-i+1)!}{(r-2(i-1))!} \qquad (1.1)$$

To ensure that there is no repetition of arrangement, we shall have (1.1) to be

$$P_{(k,i)}P_{(m,j)}P_{(N-k-m,r-i-j)}\frac{(r-i)!}{(r-i-j)!}\frac{(r-i+1)!}{(r-2(i-1))!}\frac{1}{i!}\frac{1}{j!}$$

By simplifying above expression we obtained

$$i!j!(r-i-j)!\binom{N-k-m}{r-k-i}\binom{k}{i}\binom{m}{j}\binom{r-i}{j}\binom{r-i+1}{i}\ \forall\ i<k,\ j<m \quad (1.2)$$

Now we specify the range of values for $i,\ j$ so that the prescribed conditions (k-inseparable noninclusion and m-noninclusion) are satisfied.

Observe that $k\leq r,\ m\leq r\ and\ m+k\leq r$, then clearly

$$r-k-m\geq 0 \quad \Rightarrow \quad r-i-j\ \geq\ 0\ \forall\ i<k,\ j<m$$

To ensure that the noninclusion conditions holds, we must have the range of $i,\ j$ as such $1\leq i\leq k-1,\ 0\leq j\leq m-1$. Now by applying the Second Counting Principle (SCP) over the range of $i,\ j$ in (1.2) we shall have

$$\sum_{i=1}^{k-1}\sum_{j=0}^{m-1} i!j!(r-i-j)!\binom{N-k-m}{r-i-j}\binom{k}{i}\binom{m}{j}\binom{r-i}{j}\binom{r-i+1}{i}, \quad (1.3)$$

provided condition (1) holds otherwise we shall consider

(b): Suppose $\mathbf{N-k-m<r}$.

In this case, we must choose $i,\ j$ such that

$$N-k-m\ \geq r-i-m\,;\Rightarrow i\geq r+k-N$$

$$N-k-m\ \geq r-k-j\,;\Rightarrow j\geq r+m-N$$

Thus by mere repetition of above arguments we shall have

$$\sum_{i=r+k-N}^{k-1}\ \sum_{j=r+m-N}^{m-1} i!j!(r-i-j)!\binom{N-k-m}{r-i-j}\binom{k}{i}\binom{m}{j}\binom{r-i}{j}\binom{r-i+1}{i},$$

provided condition (2) holds.

(3) $\mathbf{k\leq r}$ **and** $\mathbf{m>r}$.

By this inequalities it is easy to see that $m+k>r$ and to ensure that the prescribed condition also holds in this situation, we shall consider

(c): Suppose $\mathbf{N-k-m\geq r}$.

Observe that $m>r-k\geq 0$ by above hypothesis. Hence it is easy to see that we need at most $r-k$ elements from the set M, so that the range of the values for $i,\ j$ is such that $0\leq i\leq k-1$ and $0\leq j\leq r-k$ hence (1.3) becomes

$$\sum_{i=1}^{k-1}\sum_{j=0}^{r-k} i!j!(r-i-j)!\binom{N-k-m}{r-i-j}\binom{k}{i}\binom{m}{j}\binom{r-i}{j}\binom{r-i+1}{i}$$

provided condition (6) holds.

(d): Suppose $\mathbf{N-k-m<r}$.

In a similar manner, we need at least $r+m-N$ elements from the set M and $r+k-N$ elements from the set K, hence we have

$$\sum_{i=r+k-N}^{k-1}\ \sum_{j=r+m-N}^{r-k} i!j!(r-i-j)!\binom{N-k-m}{r-i-j}\binom{k}{i}\binom{m}{j}\binom{r-i}{j}\binom{r-i+1}{i}$$

provided condition (5) holds.

Now, we shall simply write below the range of values $i,\ j$ will take for various specifications of inequalities involving $m,\ k,\ r,\ N$ as follows;

(1) $k\leq r,\ m\leq r,\ m+k\leq r \text{ and } r+k+m\leq N$:
$(0,\,0)\leq(i,\,j)\leq(k-1,\,m-1)$

(2) $k\leq r,\ m\leq r,\ m+k\leq r \text{ and } r+k+m>N$:
$(r+k-N,\,r+m-N)\leq(i,\,j)\leq(k-1,\,m-1)$

(3) $k\leq r,\ m\leq r,\ m+k>r \text{ and } r+k+m>N$:
$(r+k-N,\,r+m-N)\leq(i,\,j)\leq(r-m,\,r-k)$

(4) $k\leq r,\ m\leq r,\ m+k>r \text{ and } r+k+m\leq N$:
$(0,\,0)\leq(i,\,j)\leq(r-m,\,r-k)$

(5) $k\leq r,\ m>r \text{ and } r+k+m>N$:
$(r+k-N,\,r+m-N)\leq(i,\,j)\leq(k-1,\,r-k)$

(6) $k\leq r,\ m>r \text{ and } r+k+m\leq N$:
$(0,\,0)\leq(i,\,j)\leq(k-1,\,r-k)$

(7) $k>r,\ m\leq r \text{ and } r+k+m>N$
$(r+k-N,\,r+m-N)\leq(i,\,j)\leq(r-m,\,m-1)$

(8) $k > r,\ m \leq r \ and\ r + k + m \leq N$

$$(0,\ 0) \leq (i,\ j) \leq (r - m,\ r - k)$$

(9) $k > r,\ m > r \ and\ r + k + m > N$

$$(r + k - N,\ r + m - N) \leq (i,\ j) \leq (k - 1,\ m - 1)$$

(10) $k > r,\ m > r \ and\ r + k + m \leq N$

$$(0,\ 0) \leq (i,\ j) \leq (k - 1,\ m - 1)$$

Hence we can denote the range of values for $(i,\ j)$ in a general form as $(\alpha_1(N, r, k, m),\ \alpha_2(N, r, k, m)) \leq (i,\ j) \leq (\beta_1(N, r, k, m),\ \beta_2(N, r, k, m))$

This complete the proof.

Theorem 2.2

Let $X = \{x_i : i = 1, 2, ..., N\}$,then the r-permutation of N distinct elements $(r \leq N)$ with the k-separable noninclusion and m-noninclusion all at a time is

$$C_{(n,r,k,m)} = \begin{cases} \displaystyle\sum_{i=\alpha_1(N,r,k,m)}^{\beta_1(N,r,k,m)} \sum_{j=\alpha_2(N,r,k,m)}^{\beta_2(N,r,k,m)} \binom{N-k-m}{r-i-j}\binom{k}{i}\binom{m}{j}, & i,\ j \neq 0 \\ \displaystyle\binom{N-k-m}{r}, & i,\ j = 0 \end{cases}$$

Proof

If order of arrangement is not important, we shall basically have the $(r - i - j)!$ factor to be equivalent to one form of arrangement, since it constituents are of the same form, hence it has a numerical value of 1. Similarly we have the same result for the $i!$,$j!$ factors.Hence the result follows immediately.

Theorem 2.3

Let $Y = \cup_{i=1}^{k} A_j,\ Z = \cup_{l=1}^{m} B_l \ and\ Y \cup Z \subset X$ such that $A_j \cap A_{j+1} = \emptyset,\ B_l \cap B_{l+1} = \emptyset\ \forall\ j,\ l\ and\ Y \cap Z = \emptyset$, then the r-permutation of N distinct elements of X with $n(Y)$-separable noninclusion and $n(Z)$-noninclusion all at a time is $P_{(n,r,k^*,m^*)}$

$$= \begin{cases} \displaystyle\sum_{i=\alpha_1(N,r,k,m)}^{\beta_1(N,r,k,m)} \sum_{j=\alpha_2(N,r,k,m)}^{\beta_2(N,r,k,m)} i!j!(r-i-j)! \binom{N - \sum_{s=1}^{k} n(A_s) - \sum_{l=1}^{m} n(B_l)}{r-i-j} \binom{\sum_{s=1}^{k} n(A_s)}{i} \\ \displaystyle\binom{\sum_{l=1}^{m} n(B_l)}{j} \binom{r-i}{j} \binom{r-i+1}{i}\ i, j \neq 0 \\ \displaystyle\binom{N - \sum_{s=1}^{k} n(A_s) - \sum_{l=1}^{m} n(B_l)}{r}, \quad i, j = 0 \end{cases}$$

Proof

There is no loss of generality if we suppose that the $A_s{}'s\ and\ B_l{}'s$ are singletons $\forall\ i,\ l$.

Thus $A_s = \{y_s\}\ and\ B_l = \{z_l\}\ \forall\ j,\ l$.

This implies

$$Y = \bigcup_{s=1}^{k} \{y_s\}\ and\ Z = \bigcup_{l=1}^{m} \{z_l\}$$

So that the cardinality

$$n(Y) = \sum_{s=1}^{k} n\left(\{y_s\}\right) = k\ and\ n(Z) = \sum_{l=1}^{m} n\left(\{z_j\}\right) = m$$

which reduces to the case of theorem 2.1. Otherwise if we suppose that $A_s{}'s\ and\ B_l{}'s$ are not singletons for every $s,\ l$, then there exist a unique numbers $k^*,\ m^* \in \mathbb{N}$ such that

$$n(Y) = \sum_{s=1}^{k} n\left(A_s\right) = k^*\ and\ n(Z) = \sum_{l=1}^{m} n\left(B_j\right) = m^*$$

Hence by theorem 2.1 the result follows immediately.

Theorem 2.4

Let $Y = \cup_{i=1}^{k} A_j,\ Z = \cup_{l=1}^{m} B_l\ and\ Y \cup Z \subset X$ such that $A_j \cap A_{j+1} = \emptyset,\ B_l \cap B_{l+1} = \emptyset\ \forall\ j,\ l\ and\ Y \cap Z = \emptyset$, then the r-combination of N distinct elements of X with $n(Y)$-separable noninclusion and $n(Z)$-noninclusion all at a time is $C_{(n,r,k^*,m^*)}$

$$= \begin{cases} \displaystyle\sum_{i=\alpha_1(N,r,k,m)}^{\beta_1(N,r,k,m)} \sum_{j=\alpha_2(N,r,k,m)}^{\beta_2(N,r,k,m)} \binom{N - \sum_{s=1}^{k} n(A_s) - \sum_{l=1}^{m} n(B_l)}{r-i-j} \binom{\sum_{s=1}^{k} n(A_s)}{i} \binom{\sum_{l=1}^{m} n(B_l)}{j} & i,j \neq 0 \\ \displaystyle\binom{N - \sum_{s=1}^{k} n(A_s) - \sum_{l=1}^{m} n(B_l)}{r},\ i,j = 0 \end{cases}$$

The proof of this result follows from theorem 5.1 and theorem 5.3

Suppose we relax (weaken) the condition imposed on $A_j{}'s\ and\ B_l{}'s$ sets,that is if $A_s{}'s\ and\ B_l{}'s$ are not necessarily (pairwise) disjoint sets then we shall consequently have the following theorems to take care of the situations.

Theorem 2.5

Let $Y = \cup_{i=1}^{k} A_j,\ Z = \cup_{l=1}^{m} B_l\ and\ Y \cup Z \subset X$ such that $Y \cap Z = \emptyset$, then the r-permutation of N distinct elements of X with $n(Y)$-inseparable noninclusion and $n(Z)$-noninclusion all at a time is $P_{(n,r,k^*,m^*)}$

$$
= \begin{cases} \displaystyle\sum_{i=\alpha_1(N,r,k,m)}^{\beta_1(N,r,k,m)} \sum_{j=\alpha_2(N,r,k,m)}^{\beta_2(N,r,k,m)} i!j!(r-i-j)! \binom{N - \sum_{\substack{I\subset[k] \\ I\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{s\in I} A_s\right) - \sum_{\substack{L\subset[m] \\ L\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{l\in L} B_l\right)}{r-i-j} \\ \displaystyle\binom{\sum_{\substack{I\subset[k] \\ I\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{s\in I} A_s\right)}{i} \binom{\sum_{\substack{L\subset[m] \\ L\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{l\in L} B_l\right)}{j} \binom{r-i}{j}\binom{r-i+1}{i}, \ i,j\neq 0 \\ \displaystyle\binom{N - \sum_{\substack{I\subset[k] \\ I\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{s\in I} A_s\right) - \sum_{\substack{L\subset[m] \\ L\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{l\in L} B_l\right)}{r}, \ i,j=0 \end{cases}
$$

To prove this theorem, first it is easy to establish that equality hold in lemma 5.0 by the use of Pascal's identity. Hence we obtain the specific values for k^* *and* m^*. Similarly the result follows immediately.

In the sequel we shall be stating results which are based on the modification(s) of condition(s) in the previous theories. Consequently, it is possible to derive several corollaries, we leave this as a simple exercise to the reader

Theorem 2.6

Let $Y = \cup_{i=1}^{k} A_j$, $Z = \cup_{l=1}^{m} B_l$ *and* $Y \cup Z \subset X$ such that $Y \cap Z = \emptyset$, then the r-combination of N distinct elements of X with $n(Y)$-inseparable inclusion and $n(Z)$-noninclusion all at a time is $c_{(n,r,k^*,m^*)}$

$$
= \begin{cases} \displaystyle\sum_{i=\alpha_1(N,r,k,m)}^{\beta_1(N,r,k,m)} \sum_{j=\alpha_2(N,r,k,m)}^{\beta_2(N,r,k,m)} \binom{N - \sum_{\substack{I\subset[k] \\ I\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{s\in I} A_s\right) - \sum_{\substack{L\subset[m] \\ L\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{l\in L} B_l\right)}{r-i-j} \\ \displaystyle\binom{\sum_{\substack{I\subset[k] \\ I\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{s\in I} A_s\right)}{i} \binom{\sum_{\substack{L\subset[m] \\ L\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{l\in L} B_l\right)}{j}, \ i,j\neq 0 \\ \displaystyle\binom{N - \sum_{\substack{I\subset[k] \\ I\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{s\in I} A_s\right) - \sum_{\substack{L\subset[m] \\ L\neq\emptyset}} (-1)^{n(I)-1} n\left(\bigcap_{l\in L} B_l\right)}{r}, \ i,j=0 \end{cases}
$$

To prove this theorem, first it is easy to establish that equality hold in lemma 5.0 by the use of Pascal's identity. Hence we obtain the specific values for k^* *and* m^*. Similarly the result follows immediately from theorem 5.2 and theorem 5.4.

For illustrative purpose we consider the following example.

Example

In a trade-fair, a Toy company with Toys of types t_i $\{i = 1,\ 2,\ \cdots 8\}$ wish to display in row five of her products so that each display is ordered uniquely. However, they required that type $t_1,\ t_2,\ t_3$ Toys must not be next to each other and will not include all at once in any display, finally $t_4,\ t_5,\ t_6$ Toys will not be included all at once. For effective cost projection, the firm require to know the number of possible display at her disposal since each display attracts a cost.

Solution

To solve this problem, it is possible to use the manual listing method of all possible display, that is the permutation of eight toys taken five at a time satisfying the prescribed conditions on $t_1,\ t_2,\ t_3$ and $t_4,\ t_5,\ t_6$. From the computation we did using the manual list approach we got 3024 displays. We do not intend generating a pictorial display of the 3024 permutations of the toys satisfying the prescribed conditions, however we leave it as an exercise for the reader to generate.

Now by the formula method if we take $M = \{t_4,\ t_5,\ t_6\}$ *and* $K = \{t_1,\ t_2,\ t_3\}$ so that $k = 3$, $m = 3$, $r = 5$ & $N = 8$, hence we have

$$\sum_{i=1}^{2}\sum_{j=0}^{2} i!j!(5-i-j)!\binom{2}{5-i-j}\binom{3}{i}\binom{3}{j}\binom{5-i}{j}\binom{6-i}{i} = 3024$$

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