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On Uniformly Continuous Uniformity On A Topological Space

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ABSTRACT:

Given a topological space $(X,\mathcal{T})$ we define uniformity $\mathcal{U}$ on X such that $(X,\mathcal{U})$ is uniformly continuous uniform space and it becomes the smallest uniformly continuous uniformity with respect to which set of continuous functions is larger than set of $\mathcal{T}$ –continuous functions.

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INTRODUCTION

Let X be any non empty set. Let $\mathcal{P}$ be the family of all pseudo metrics on X. Then $\mathcal{P} \neq \emptyset$ since discrete metric on X is a pseudo metric. For any subfamily $\mathcal{Q}$ of $\mathcal{P}$ define the uniformity on X whose subbase is the family of all sets

$$V_{p,r} = \{(x,y) \in XxX / p(x,y)<r\},\ p \in \mathcal{Q} \text{ and } r > 0$$

$\therefore \mathcal{B} = \{B \subset X \times X / B = \bigcap_{i=1}^{n} V_{p_i,r_i}, p_i \in \mathcal{P}, r_i > 0, n \geq 1\}$ is a base for uniformity on X.

Now let $(X,\mathcal{T})$ be a topological space. Let $\mathcal{C}(X)$ be the set of all continuous complex valued functions on X. For every $f \in \mathcal{C}(X)$, define $d_f : XxX \to \mathbb{R}$ as $d_f(x,y) = | f(x) - f(y) |$. Then d_f is a pseudo -metric on X. With the help of pseudo-metric d_f on X define an open sphere $S_r(x)$, $x \in X$, $r > 0$ as $S_r(x) = \{ y: d_f(x,y) < r \}$

As the intersection of any two open spheres contains an open sphere about each of its pts , the family $\{ S_r(x) / x \in X \}$ is a base for topology for X. This topology is the pseudo-metric topology for X generated by pseudo-metric d_f on X say $\mathcal{T}_{d_f}$.

Let P be the family of pseudo metrics d_f on X as $f \in \mathcal{C}(X)$. Then P defines a unique uniformity on X such that $S = \{ V_{f,r} / f \in \mathcal{C}(X) \text{ and } r > 0 \}$ forms a subbase where $V_{f,r} = \{(x,y) \in XxX / d_f(x,y) < r\}$ We denote this uniformity on X by $\mathcal{U}$.

Definition: Uniformly continuous uniform space:

A uniform space is said to be Uniformly continuous uniform space if every real valued continuous function is uniformly continuous.

Theorem1: Let $(X,\mathcal{T})$ be a topological space and $\mathcal{U}$ be the uniformity defined on X as above then $(X,\mathcal{U})$ is a Uniformly continuous space.

Proof: For proving this theorem we require following two lemmas.

Lemma1. Every $\mathcal{U}$ -continuous mapping is $\mathcal{T}$ -continuous.

Proof: Let $f: X \to \mathbb{C}$ be any $\mathcal{U}$ -continuous mapping, where $\mathcal{U}$ is the above defined uniformity on X. Then we show that f is $\mathcal{T}$ -continuous. Let G be any open set in $\mathbb{C}$. Then $f^{-1}(G)$ is $\mathcal{U}$– open in X. i.e. $f^{-1}(G) \in \mathcal{T}_{\mathcal{U}}$ where $\mathcal{T}_{\mathcal{U}} = \{T \subset X / \forall x \in T\ \exists\ U \in \mathcal{U} \text{ s.t. } U[x] \subset T\}$. Now we show that $\mathcal{T}_{\mathcal{U}} \subseteq \mathcal{T}$.

Suppose T $\epsilon\ \mathcal{T}_{\mathcal{U}}$ and $x\ \epsilon$ T. Then we have U $\epsilon\ \mathcal{U}$ s.t. U[x]⊂ T. Since U $\epsilon\ \mathcal{U}$ by def^n of $\mathcal{U}$ ∃ $f_1,f_2,\ldots..f_n \in \mathcal{C}(X)$ and $r_1,r_2,\ldots\ldots.r_n$ all positive such that $V=\bigcap_{i=1}^{n} V_{f_i,r_i} \subset U$. Then $V[x] \subset U[x] \subset T$.

But V[x] = { y: (x,y) $\epsilon\ V$ }

$=\{$ y: (x,y) $\epsilon \bigcap_{i=1}^{n} V_{f_i,r_i}$ }

$=\{$ y: (x,y) $\epsilon\ V_{f_i,r_i}\ \forall i=1,2,\ldots\ldots n$ }

$=\{$ y: $f_i(y) \in S(f_i(x),r_i)\ \ \forall i=1,2,\ldots\ldots n$ } }

$=\{$ y: y $\epsilon\ f_i^{-1}(S(f_i(x),r_i))\ \ \forall i=1,2,\ldots\ldots n$ } }

$=\{$ y: y $\epsilon \bigcap_{i=1}^{n} f_i^{-1}(S(f_i(x),r_i))$)

Now for each i, $(S(f_i(x),r_i))$ is an open subset of $\mathbb{C}$ and each f_i is $\mathcal{T}$-continuous hence $f_i^{-1}(S(f_i(x),r_i))$ is $\mathcal{T}$ – open. ∴ $\bigcap_{i=1}^{n} f_i^{-1}(S(f_i(x),r_i))$ is $\mathcal{T}$– open. *ie.* $V[x]$ is $\mathcal{T}$– open . ie. *for every* $x\ \epsilon\ T$ ∃ $\mathcal{T}$– open set V[x] such that $x\ \epsilon\ V[x] \subset T$. This proves that T $\epsilon\ \mathcal{T}$. Hence $\mathcal{T}_{\mathcal{U}} \subseteq \mathcal{T}$.

Lemma2: If f is $\mathcal{T}$ –continuous then f is $\mathcal{U}$-uniformly continuous function.

Proof: Since f is $\mathcal{T}$-continuous function, $d_f(x,y)=$ l f(x)-f(y) l ,x,y ϵ X defines a pseudo metric on X and hence $d_f\ \epsilon$ P. Then for any r >0 $U_{f,r}=\{(x,y)/d_f(x,y)<r\}$ is a subbase member of the uniformity $\mathcal{U}$. i.e. $U_{f,r}\ \epsilon\ \mathcal{U}$ such that (x,y) $\epsilon\ U_{f,r}$ ⇔ l f(x)-f(y) l < r.Thus f is $\mathcal{U}$-uniformly continuous function.

Proof of Theorem 1: Suppose f is $\mathcal{U}$ - continuous complex valued function on X. By lemma1 f is $\mathcal{T}$ – continuous. By Lemma 2 f is *then* $\mathcal{U}$ – uniformly continuous function. Hence (X, $\mathcal{U}$) is uniformly continuous uniform space.

Note: From lemma1 $\mathcal{C}_{\mathcal{T}_{\mathcal{U}}} \subset \mathcal{C}_{\mathcal{T}}$. But from lemma2 every $\mathcal{T}$–continuous function is $\mathcal{U}$-uniformly continuous and hence it is $\mathcal{U}$–continuous function.
ie. $\mathcal{C}_{\mathcal{T}} \subseteq \mathcal{C}_{\mathcal{T}_{\mathcal{U}}} \Rightarrow \mathcal{C}_{\mathcal{T}} = \mathcal{C}_{\mathcal{T}_{\mathcal{U}}}$.

Theorem2: Suppose (X,$\mathcal{V}$) is a uniformly continuous uniform space such that $\mathcal{C}_{\mathcal{T}} \subseteq \mathcal{C}_{\mathcal{T}_{\mathcal{V}}}$,then $\mathcal{U} \subset \mathcal{V}$ where $\mathcal{U}$ is the uniformity on X determined by $\mathcal{T}$–continuous functions on X. i.*e.* $\mathcal{U}$ is the smallest uniformly continuous uniform space such that $\mathcal{C}_{\mathcal{T}} \subseteq \mathcal{C}_{\mathcal{T}_{\mathcal{U}}}$.

Proof: To show that $\mathcal{U} \subset \mathcal{V}$, let U $\epsilon\ \mathcal{U}$. Then there are $\mathcal{T}$–continuous functions $f_1,\ldots.f_n$ on X and $\epsilon_i > 0$, $i=1,2,3\ldots\ldots n$ such that W = $\bigcap_{i=1}^{n}\{(x,y): \mathrm{l}\, f_i(x)-f_i(y) \mathrm{l} < \epsilon_i\} \subset$ U ……(1) .
Since each $\mathcal{T}$–continuous function f_i is $\mathcal{T}_{\mathcal{V}}$–continuous and $\mathcal{V}$ is a uniformly continuous uniform space, each f_i is $\mathcal{V}$ - uniformly continuous for $i=1,2,3\ldots..n$. Hence for each $\epsilon_i > 0\ \exists\ V_i \in \mathcal{V}$ such that $(x,y)\ \epsilon\ V_i \Rightarrow |f_i(x)-f_i(y)| < \epsilon_i$. *i.e.* $V=\bigcap_{i=1}^{n} V_i \subset W \subset U$. But $V=\bigcap_{i=1}^{n} V_i \in \mathcal{V}$ ∴ $U \in \mathcal{V}$(by definition of uniform space) i.e. $\mathcal{U} \subset \mathcal{V}$… *ie.* $\mathcal{U}$ is the smallest uniformly continuous uniform space such that $\mathcal{C}_{\mathcal{T}} \subseteq \mathcal{C}_{\mathcal{T}_{\mathcal{U}}}$.

Corollary: Suppose (X,$\mathcal{V}$) is a uniformly continuous uniform space such that $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{V}}$ then $\mathcal{U} \subset \mathcal{V}$ ….

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