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Integral Solutions of Binary Quadratic Diophantine equation $x^2 + pxy + y^2 = N$

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ABSTRACT
Non – trivial integral solutions for the binary quadratic diophantine equation $x^2 + pxy + y^2 = N$, $p > 2$, $N \not\equiv 0 \pmod 4$ *are obtained . The recurrence relations satisfied by the solutions along with a few examples are given.*

KEY WORDS: *Binary quadratic, integral solutions MSC 2000 Subject classification number: 11D09*

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I. INTRODUCTION

It is well known that binomial quadratic (homogeneous or non-homogeneous) Diophantine equations are rich in variety[1,2]. The authors have considered the equation $x^2 + xy + y^2 = N$, and analysed for its integer solutions[3]. In [4], non-trivial integral solutions for the binary quadratic diophantine equation $x^2 + pxy + y^2 = 4N$ are obtained. In this communication, the non-trivial integral solutions for the binary quadratic diophantine equation $x^2 + pxy + y^2 = N$, where $p > 2$ and $N \not\equiv 0 \pmod 4$ have been obtained. Also the recurrence relations among the solutions are given.

II. METHOD OF ANALYSIS

The equation to be solved is

$$x^2 + pxy + y^2 = N, p > 2, N \not\equiv 0 \pmod 4 \qquad (1)$$

The substitution of the linear transformations

$$\left.\begin{aligned} u &= X + (2-p)T \\ v &= X - (2+p)T \end{aligned}\right\}$$

inequation (1) leads to

$$4X^2 = (p^2 - 4)4T^2 + N \qquad (2)$$

Again, setting $2X = P,\ 2T = Q$

Equation (2) becomes

$$P^2 = (p^2 - 4)Q^2 + N \qquad (3)$$

where $p^2 - 4$ is a square free non zero integer.

Assume that the initial solution of equation (2) be $\left(Q_0, P_0\right)$.

Consider the Pellian

$$P^2 = \left(p^2 - 4\right)Q^2 + 1 \tag{4}$$

whose general solution $(\tilde{Q}_s, \tilde{P}_s)$ is given by

$$\tilde{P}_s + \sqrt{p^2 - 4}\ \tilde{Q}_s = \left(\tilde{P}_0 + \sqrt{p^2 - 4}\ \tilde{Q}_0\right)^{s+1},\ s = 0,1,2,.... \tag{5}$$

in which $\left(\tilde{Q}_0, \tilde{P}_0\right)$ is the least positive integral solution of (4).

Applying Brahmagupta's lemma, the sequence of solutions of equation (3) are given by

$$\left.\begin{aligned} Q_{s+1} &= P_0\tilde{Q}_s + Q_0\tilde{P}_s \\ P_{s+1} &= P_0\tilde{P}_s + \left(p^2 - 4\right)Q_0\tilde{Q}_s \end{aligned}\right\} \tag{6}$$

where $s = 0,1,2,........$

$\therefore$ the sequence of solutions of equation (1) are given by

$$\left.\begin{aligned} x_{s+1} &= \frac{1}{2}\left[P_0G + \sqrt{p^2 - 4}\ Q_0F\right] - \frac{p}{2\sqrt{p^2 - 4}}\left[P_0F + \sqrt{p^2 - 4}\ Q_0G\right] \\ y_{s+1} &= \frac{1}{\sqrt{p^2 - 4}}\left[P_0F + \sqrt{p^2 - 4}\ Q_0G\right] \end{aligned}\right\} \tag{7}$$

where $F = \left(\tilde{P}_0 + \sqrt{p^2 - 4}\ \tilde{Q}_0\right)^{s+1} + \left(\tilde{P}_0 - \sqrt{p^2 - 4}\ \tilde{Q}_0\right)^{s+1}$

$$G = \left(\tilde{P}_0 + \sqrt{p^2 - 4}\ \tilde{Q}_0\right)^{s+1} + \left(\tilde{P}_0 - \sqrt{p^2 - 4}\ \tilde{Q}_0\right)^{s+1},\ s = -1,0,1,2,...$$

Also the recurrence relations among the solutions are given by

(1) $x_{s+3} - 2\tilde{P}_0 x_{s+2} + x_{s+1} \equiv 0$

(2) $y_{s+3} - 2\tilde{P}_0 y_{s+2} + y_{s+1} \equiv 0$

To analyze the nature of solutions, one has to go in for particular values of p and N. For the sake of simplicity and clear understanding, a few numerical examples are given below:

Illustration :1.1

p and N are both odd.

Table 1.1(a)

$p = 3 \ \ N = 5$

i	x_i	y_i
0	-1	4
1	-29	76
2	-521	1364
3	-9349	24476
4	-167761	439204
5	-3010349	7881196
6	-54018521	141422324
7	-969323029	2537720636
8	-17393796001	45537549124
9	-312119004989	817138163596

Observations :

(1) $y_i \equiv x_i \pmod{5}$

(2) Each of the expressions $y_{2i} + x_{2i} - 2$ and $y_{2i+1} + x_{2i+1} + 2$ is a perfect square.

(3) $x_i y_{i+1} - y_i x_{i+1} \equiv 0 \pmod{40}$

Illustration :1.2

p is even and N is odd

Table 1.2 (b)

$p = 4 \ \ N = 13$

i	x_i	y_i
0	1	2
1	-9	34
2	-127	474
3	-1769	6602
4	-24639	91954
5	-343177	1280754
6	-4779839	17838602
7	-66574569	248459674
8	-927264127	3460596834
9	-12915123209	48199896002

Observations :

(1) $y_{i+1} \equiv y_i \pmod 8$

(2) $y_{3i-1} - y_{3i-2} \equiv 0 \pmod{10}$

(3) $x_i + y_i \equiv x_i - y_i \pmod 4$

(4) $x_{i-1} \equiv x_i \pmod 2$

(5) $x_{i+1}y_i - y_{i+1}x_i + 52 \equiv 0$

Illustration :1.3

p and N are both even.

Table 1.3(c)

$$p = 4 \ N = 4$$

i	x_i	y_i
0	0	2
1	-8	30
2	-112	418
3	-1560	5822
4	-21728	81090
5	-302612	1129438
6	-4214840	15731042
7	-58705148	219105150
8	-817657232	3051741058
9	-11388496100	42505269662

Observations :

(1) $x_{i+1}y_i - y_{i+1}x_i \equiv 0 \pmod{16}$

(2) $x_i + y_i \equiv x_i - y_i \pmod 4$

(3) $y_{i+1} \equiv y_i \pmod 4$

(4) $y_{3i-1} \equiv y_{3i-2} \pmod 4$

Illustration :1.4

p is odd and N is even.

Table 1.4(d)

$p = 3 \;\; N = 4$

i	x_i	y_i
0	-2	6
1	-54	110
2	-970	1974
3	-17406	35422
4	-312338	635622
5	-5604678	11405774
6	-100571866	204668310
7	-1804688910	3672623806
8	-32383828514	65902560198
9	-581104224342	1182573459758

Observations:

(1) $x_{i+1}y_i - y_{i+1} = -104$

(2) $y_i - x_i \equiv y_i + x_i \pmod{4}$

In conclusion, one may search for other patterns of solutions and their corresponding properties.

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