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Homotopy Analysis Method for One-dimensional Heat Conduction in A Bar with Temperature-dependent Thermal Conductivity.

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Abstract

The Homotopy Analysis Method (HAM) is employed in the analysis of non-linear one-dimensional heat conduction problem with temperature-dependent thermal conductivity. The temperature distribution is obtained in terms of the scale space variable x and as a function of B which is a parameter in the heat conduction equation. It is observed that the parameter B has a strong influence over the rate of heat conduction in the bar. By choosing the convergence parameter, h, in a suitable way, we obtained solutions for the temperature distribution for various values of B. From this temperature distribution other heat transfer quantities can be obtained. It is observed that for some values of B the temperature distribution in the bar is an increasing function of B while for other values of B it a decreasing function. The results are displayed in figures.

Keywords: Homotopy Analysis Method, Temperature-dependent Thermal Conductivity, Heat Conduction.

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Nomenclature

A_c	cross-sectional area of the material
X	dimensional space coordinate
D_m	mth-order Homotopy-derivative
K	dimensional thermal conductivity
k	dimensionless thermal conductivity
L	length of the metal bar
$\mathcal{L}$	auxiliary linear operator
T	temperature
x	dimensionless space coordinate
h	auxiliary parameter
$H(\xi)$	auxiliary function
q	embedding
U	dimensionless temperature
B	parameter in equation (3)
Subscripts	
a	ambient condition
b	end of the bar
m	mth order of approximation
Greeks	
χ_m	two-valued function
ξ	similarity variables

I. INTRODUCTION

Nonlinear phenomena play a crucial role in applied mathematics, engineering and physics. The theory of nonlinear problems has recently undergone many studies. One of the nonlinear problems encountered in real life is the heat conduction in a bar with temperature dependent thermal conductivity. The thermal conductivity of a material generally varies with temperature. However, this variation is mild for many materials in the range of practical interest and can be disregarded. Even in such cases, an average value for the thermal conductivity is used and treated as constant. When the variation of thermal conductivity with temperature in a specified temperature interval is large; however, it becomes necessary to account for this variation to minimize error. Accounting for the variation of the thermal conductivity with temperature, in general complicates the analysis. Due to non-linearity of the problem, in most cases numerical approaches are common but analytical approaches seem to be intractable. The basic idea of Homotopy in topology provided an idea to propose a general analytic method for non-linear problems namely homotopy analysis method (HAM).The method was proposed by Liao S.J. [8] in 1992. This method is now widely used to solve many types of nonlinear problems. For example, Khani, F. et al, [7] used HAM for the solutions and efficiency of the non-linear fin problem in which he showed the efficiency of HAM. Abbasbandy, S. [3] applied HAM to solve a generalized Hirota-Satsuma coupled KdV equation. Liao et al [5] applied HAM for the solutions of the Blasius viscous flow problems. Other researchers have since employed HAM for solutions of seeming difficult scientific and engineering problems. In this paper, we introduce the basic idea of HAM and apply it to find an approximate analytical solution of the nonlinear one-dimensional heat conduction problem with temperature dependent thermal conductivity.

II. MATHEMATICAL FORMULATION

Here we consider a straight one-dimensional metal bar with a constant cross-sectional area, A_c. The bar with length, L, is exposed to a temperature source T_b and extends into a fluid at ambient temperature T_a. We assume that the conduction heat transfer is in a steady-state and no heat is generated within the bar. The one-dimensional steady state heat balance equation in dimensional form may be considered as

$$A_c \frac{d}{dX}\left(K\frac{dT}{dX}\right) = 0, \quad 0 < X < L$$

Where K is the thermal conductivity of the metal bar and in many engineering applications it may be assumed to be a linear function of temperature

$$K = K(T) = K_a[1 + \lambda(T - T_a)]. \qquad (2)$$

Here, K_a is the thermal conductivity of the metal at the ambient temperature and λ is the parameter describing the dependence on the temperature and is called the temperature coefficient of thermal conductivity. By introducing the following dimensionless variables

$$U = \frac{T - T_a}{T_b - T_a}, \quad x = \frac{X}{L}, \quad k = \frac{K}{K_a}, \quad B = \lambda(T_b - T_a) \qquad (3)$$

the general equation becomes

$$\frac{d}{dx}\left(k(U)\frac{dU}{dx}\right) = 0, \quad 0 < x < L, \qquad (4)$$

Which is subject to

$$U'(0) = 0 \ \text{at}\ x = 0$$
$$U(1) = 0 \ \text{at}\ x = L \qquad (5)$$

The heat balance equation (4) can be rewritten as

$$U'' + BU''U + BU'^2 = 0 \qquad (6)$$

In general, the analytic solution of Equation (6) is difficult. As a matter of fact, only if $B = 0$ will this equation posses an analytical solution. In engineering applications, an explicit approximation with an acceptable accuracy is preferred rather than an exact result in an implicit form with a rigorous proof. As a result, this paper intends to look for a solution algorithm which can generate an explicit approximate solution in terms of ordinary functions.

III. A REVIEW OF HAM

The basic idea of HAM which was introduced by Liao [8] is needed to deal with the goal of this work. The following definitions are needed for the start.

Definition 1. Let ϕ be a function of the Homotopy-parameter q. Then

$$D_m = \frac{1}{m!}\frac{\partial^m \phi}{\partial q^m}\bigg|_{q=0} \qquad (7)$$

is called the mth-order Homotopy-derivative of ϕ, where $m \geq 0$ is an integer.

Let us consider the following nonlinear equation in a general form:

$$\mathcal{N}[u(\xi)] = 0$$

where $\mathcal{N}$ is a nonlinear operator, $u(\xi)$ is an unknown function, and is temporal independent variables. For simplicity, we ignore all boundary or initial conditions, which can be treated in a similar way.
This is a generalization of the traditional Homotopy method; Liao [8] in his paper constructs the so called zero-order deformation equation

$$(1-q)\mathcal{L}[\phi(\xi:q) - u_0(\xi)] = qhH(\xi)\mathcal{N}[\phi(\xi:q)], \quad (8)$$

where $q \in [0,1]$ is the embedding parameter, h is non-zero and is called the convergence parameter, $H(\xi)$ is an auxiliary function, $\mathcal{L}$ is an auxiliary operator, $u_0(\xi)$ is an initial guess for $u(\xi)$, and is an unknown function. A great freedom in choosing the auxiliary variables in Ham is of paramount importance. It is clear that when $q = 0$ and $= 1$, we have

$$\phi(\xi:0) = u_0(\xi) \text{ and } \phi(\xi:1) = u(\xi).$$

Thus, as q increases from 0 to 1, the solution $\phi(\xi:q)$ varies from the initial guess $u_0(\xi)$ to the solution $u(\xi)$. Expanding $\phi(\xi:q)$ in Taylor's series with respect to q, gives

$$\phi(\xi:q) = u_0(\xi) + \sum_{m=1}^{+\infty} u_m(\xi)q^m \quad (9)$$

where

$$u_m(\xi) = D_m[\phi(\xi:q)]. \quad (10)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter, and the auxiliary function are properly chosen, the series (9) converges at $q = 1$ and we get,

$$u(\xi) = u_0(\xi) + \textstyle\sum_{m=1}^{+\infty} u_m(\xi).$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao[8]. For $h = -1$and $H(\xi) = -1$, equation (8) becomes

$$(1-q)\mathcal{L}[\phi(\xi;q) - u_0(\xi)] - q\mathcal{N}[\phi(\xi;q)] = 0,$$

while the solution is obtained directly and without using Taylor's series. When $H(\xi) = 1$ Equation (8) changes to

$$(1-q)\mathcal{L}[\phi(\xi;q) - u_0(\xi)] = qh\mathcal{N}[\phi(\xi;q)], \quad (11)$$

According to definition (10), the governing equation can be deduced from the zero-order deformation Equation (8),
Define the vector

$$\vec{u}_m = \{u_0(\xi), u_1(\xi), \ldots, u_n(\xi)\}.$$

Operating on both side of Equation (8) byD_m, we have the so called mth-order deformation equation

$$\mathcal{L}[u_m(\xi) - \mathcal{Z}_m u_{m-1}(\xi)] = hH(\xi)R_m(\vec{u}_{m-1}, \xi) \quad (12)$$

where

$$R_m(\vec{u}_{m-1}, \xi) = D_{m-1}([\phi(\xi;q)]), \quad (13)$$

and

$$\mathcal{Z}_m = \begin{cases} 0, & m \le 1. \\ 1, & m \ge 2. \end{cases}$$

Substituting (9) into (13), we have

$$R_m(\vec{u}_{m-1}, \xi) = D_{m-1}\left(\mathcal{N}[\textstyle\sum_{n=0}^{+\infty} u_n(\xi)q^n]|_{q=0}\right) \quad (14)$$

It is of paramount importance to emphasize that $u_m(\xi)$ for $m \ge 1$ is governed by the linear Equation (12) with the linear boundary conditions which comes from the original problem and can be easily solved by symbolic computation software, like Mathematical, Maple, and Matlab.

IV. SOLUTION FOR THE TEMPERATURE DISTRIBUTION

In this section, we obtain analytical approximate solutions of Equation (6), and to apply HAM to Equation (6), it is necessary to introduce the Molabahrami and Khani's Theorems. Molabahrami and Khani [6] proved the following theorems.

Theorem 1. For Homotopy-series $\phi = \sum_{n=0}^{+\infty} U_n q^n$, it holds that

$$D_m(\phi^k) = \sum_{r_1=0}^{m} U_{m-r_1} \sum_{r_2=0}^{r_1} U_{r_1-r_2} \sum_{r_3=0}^{r_2} U_{r_2-r_3} \cdots \sum_{r_{k-2}=0}^{r_{k-3}} U_{r_{k-3}-r_{k-2}} \sum_{r_{k-1}=0}^{r_{k-2}} U_{r_{k-2}-r_{k-1}} U_{r_{k-1}},$$

where$m \ge 0$ and $k = 0$ are positive integers.
Proof: See [6]

Corollary 1.from theorem 1, we have

$$D_m(\phi^{k-1}\phi') = \sum_{r_1=0}^{m} w_{m-r_1} \sum_{r_2=0}^{r_1} w_{r_1-r_2} \sum_{r_3=0}^{r_2} w_{r_2-r_3} \cdots \sum_{r_{k-2}=0}^{r_{k-3}} w_{r_{k-3}-r_{k-2}} \sum_{r_{k-1}=0}^{r_{k-2}} w_{r_{k-2}-r_{k-1}} w'_{r_{k-1}},$$

and

$$D_m(\phi^{k-1}\phi") = \sum_{r_1=0}^{m} w_{m-r_1} \sum_{r_2=0}^{r_1} w_{r_1-r_2} \sum_{r_3=0}^{r_2} w_{r_2-r_3} \cdots \sum_{r_{k-2}=0}^{r_{k-3}} w_{r_{k-3}-r_{k-2}} \sum_{r_{k-1}=0}^{r_{k-2}} w_{r_{k-2}-r_{k-1}} w"_{r_{k-1}},$$

from initial conditions (5), it is reasonable to express the solution by the set of base functions

$$\{e_n(x|n \geq 0)\}, \qquad (15)$$

in the form

$$U(x) = \sum_{n=0}^{+\infty} a_n e_n(x), \qquad (16)$$

wherea_n is a coefficient and $e_n(x)$ is determined to provide the so-called rule of solution of expression $u(x)$. Under the rule of solution denoted by (16), and from Equation (6), it is straightforward to choose

$$\mathcal{L}[\phi(x:q)] = \frac{\partial^2\phi(x:q)}{\partial x^2} \qquad (17)$$

as an auxiliary linear operator which has the property

$$\mathcal{L}[c_0 + c_1 x] = 0$$

where c_0 and c_1 are coefficients. The solution given by HAM denoted by (16) can be represented by many different base functions. In this work, we set $e_n(x) = x^{2n}$. According to Eq. (6), we define the non-linear operator

$$\mathcal{N}[\phi(x;q)] = \frac{\partial^2\phi(x:q)}{\partial x^2} + B\phi(x;q)\frac{\partial^2\phi(x:q)}{\partial x^2} + B\left(\frac{\partial\phi(x:q)}{\partial x}\right)^2 \qquad (18)$$

From Equations (13) and (18) and Theorem 1 and Corollary 1, we have

$$R_m[\vec{U}_{m-1}(x)] = D_{m-1}(\phi) + BD_{m-1}(\phi\phi") + BD_{m-1}((\phi)^2).$$

From the above equation, the following results are achieved:

$$R_1[\vec{U}_0(x)] = U_0 + BU_0U_0" + B(U_0')^2,$$

$$R_2[\vec{U}_1(x)] = U_1" + B(U_0U_1" + U_1U_0") + B(2U_0'U_1'),$$

$$R_3[\vec{U}_2(x)] = U_2" + B(U_0U_2" + U_1U_1" + U_2U_0") + B(2U_0'U_1' + (U_1')^2).$$

etc., where the primes denote differentiation with respect to x.

Now, the solution of the mth-order deformation Equation (11) under Equation (18) with initial conditions $U'_m(0) = 0$ and $U_m(1) = 0$, for $m \geq 1$, is

$$U_m(x) = \chi_m U_{m-1}(x) + h\int_0^x dw \int_0^w H(v) R_m\left(\vec{U}_{m-1}(v)\right) dv + c_1 x + c_0. \qquad (19)$$

According to Equations. (6) and (5) and the rule of solution expression (16), it is straightforward to show that the initial approximation can be written in the form

$$U_0(x) = x. \qquad (20)$$

According to the rule of expression denoted by (16) and from Equation (19) the auxiliary function $H(x)$, should be as follows;

$$H(x) = 1$$

Now from Equation (19), we can successively obtain

$$U_1(x) = -\frac{h}{6} - \frac{Bh}{2} + \frac{1}{2}Bh\,x^2 + \frac{hx^3}{6}$$

$$U_2(x) = -\frac{h}{6} - \frac{Bh}{2} - \frac{h^2}{6} - \frac{2Bh^2}{3} - \frac{B^2h^2}{2} + \frac{1}{2}Bh^2x^2 + \frac{hx^3}{6} + \frac{1}{2}B^2h^2x^3 + \frac{1}{6}Bh^2x^4$$

$$U_3(x) = \frac{-h}{6} - \frac{Bh}{2} - \frac{h^2}{3} - \frac{4Bh^2}{3} - B^2h^2 - \frac{h^3}{6} - \frac{53Bh^3}{72} - \frac{41B^2h^3}{60} - \frac{B^3h^3}{8} + \frac{1}{2}Bhx^2 + Bh^2x^2 + \frac{1}{2}Bh^3x^2$$
$$- \frac{1}{12}B^2h^3x^2 - \frac{1}{4}B^3h^3x^2 + \frac{hx^3}{6} + \frac{h^2x^3}{3} + B^2h^2x^3 + \frac{h^3x^3}{6} - \frac{1}{36}Bh^3x^3 + \frac{7}{12}B^2h^3x^3 + \frac{1}{4}Bh^3x^4 + \frac{11}{60}B^2h^3x^5$$
$$+ \frac{1}{72}Bh^3x^6$$

⋮ ⋮

In this way, we derive $U_m(x)$ for $m = 1,2,3$………. successively .At the kth –order approximation, we have the analytic solution of Equation (6), namely

$$U(x) \approx \sum_{m=0}^{k} U_m(x) . \qquad (21)$$

Liao [8] presented the auxiliary parameter in order to control the rate of convergence of the approximate solutions obtained by HAM. The parameter h determines the convergence region and the rate of approximation for HAM. For this purpose, $h-$ curves are displayed in Figures 1-3. It is clear from these figures that for values of $B < 0$ and $B \geq 1$, $h = -1$ fall within the convergence region, i.e., that HAM is convergent for these values of B. Considering figures 1-3, it is further observed that h converges at ($-1.8 \leq h \leq -0.4$), therefore taking h= -0.5 enables the plotting of U(x) against x for various values of B in figures 4-12.

V. RESULTS AND DISCUSSION

From figure 4, it seen that the temperature distribution when (B=0) is a linear function of x. This is the simplest expression of the temperature distribution in the bar.

Figures 5, 6, 8 and 12 illustrate the temperature distribution for various values of B. It is clear from the figures that the temperature distribution in the bar is an increasing function of B, while in figures 7, 9, 10 and 11 the temperature distribution in the bar is a decreasing function of B.

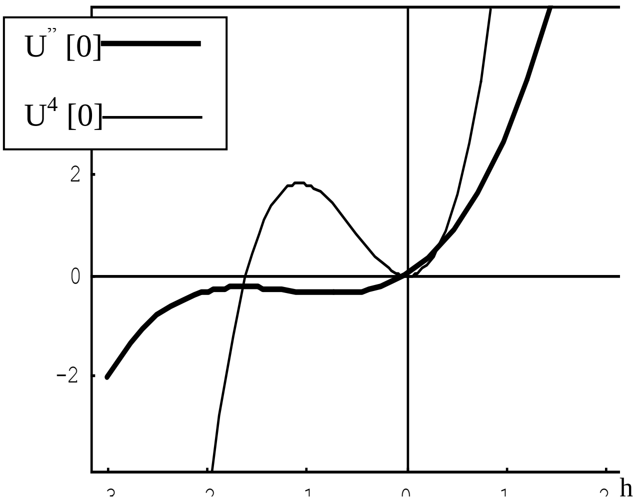

Figure 1. The h-curves for B=0.4, for 4th-order approximation of U'' [0] and U^4 [0].

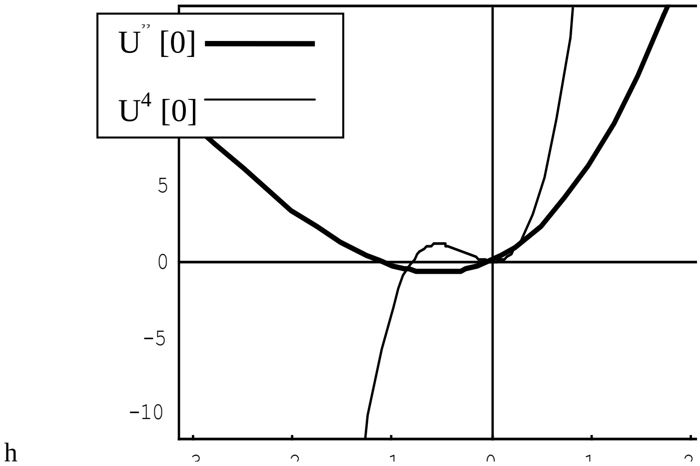

Figure 2. The h-curves for B=1.0, for 4th-order approximation of U" [0] and U^4 [0].

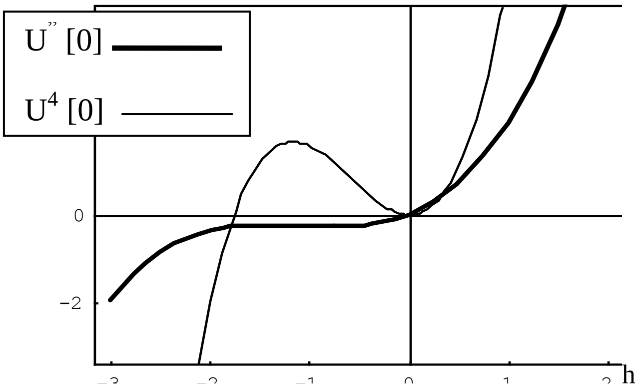

Figure 3. The h-curves for B= 0.3, for 4th-order approximation of U" [0] and U^4 [0].

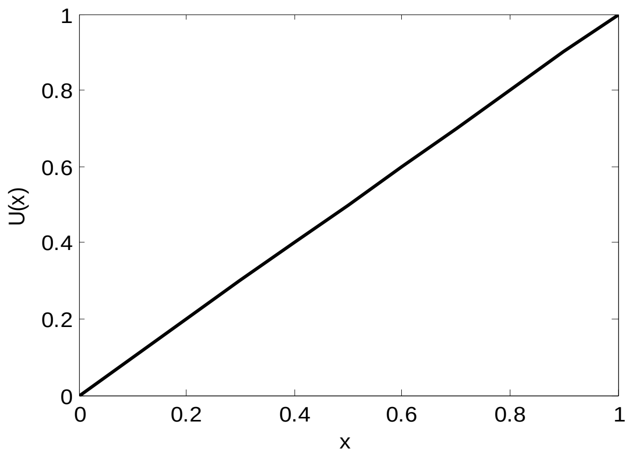

Figure 4. Temperature distribution for B= 0.

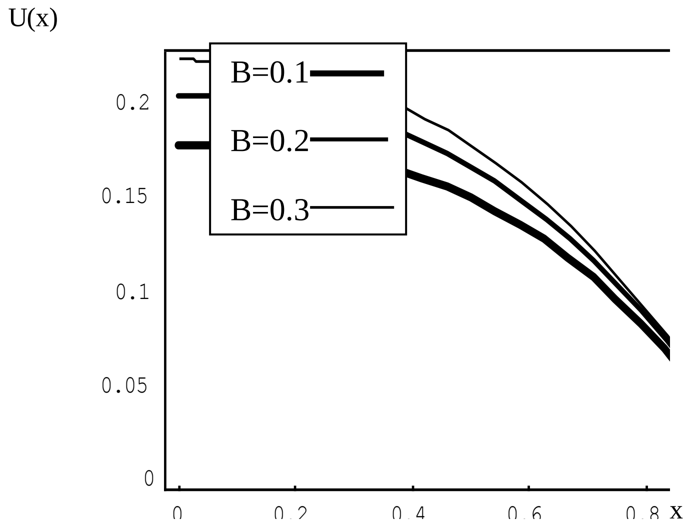

Figure 5. Temperature distribution for B= 0.1, 0.2, 0.3.

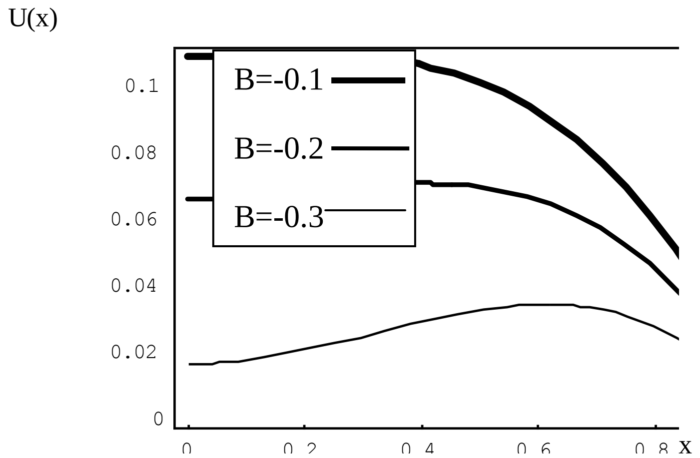

Figure 6. Temperature distribution for B= -0.1, -0.2, - 0.3

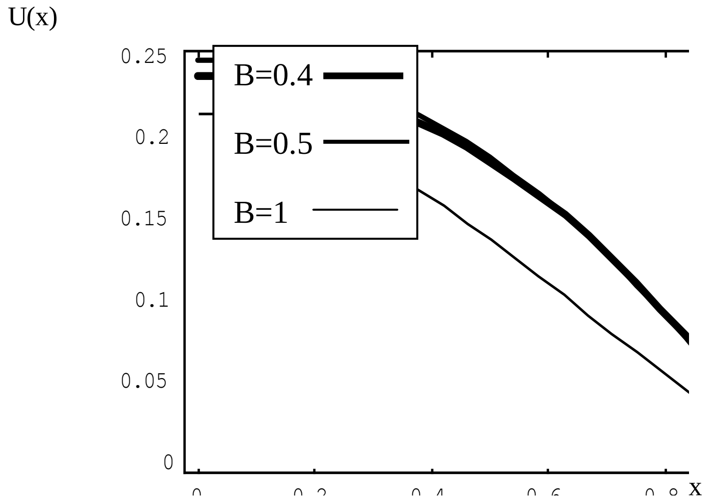

Figure 7. Temperature distribution for B= 0.4, 0.5, 1.

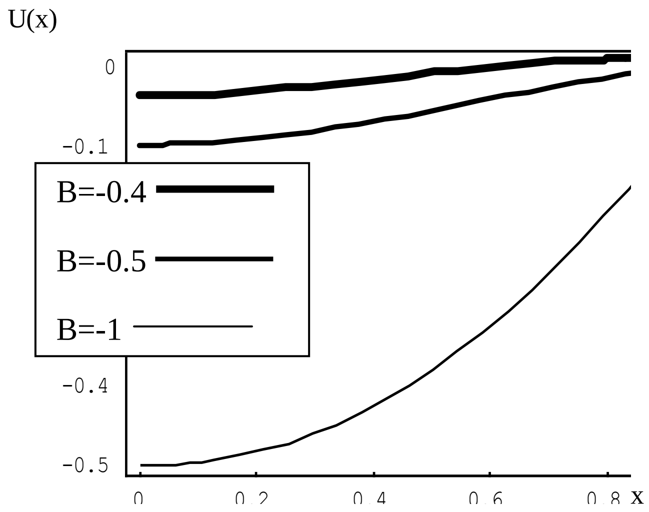

Figure 8. Temperature distribution for B= -0.4, -0.5, -1.

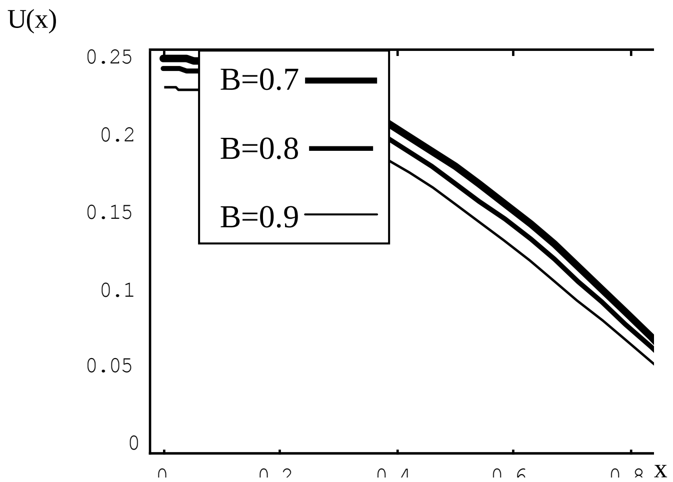

Figure 9. Temperature distribution for B= 0.7, 0.8, 0.9.

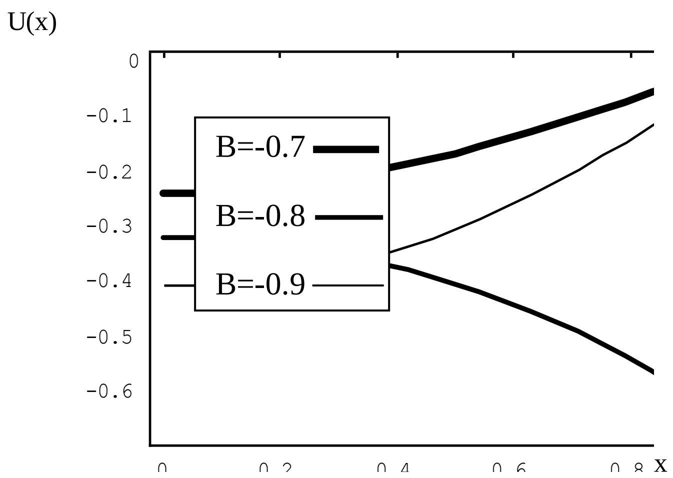

Figure 10. Temperature distribution for B= -0.7, -0.8, -0.9.

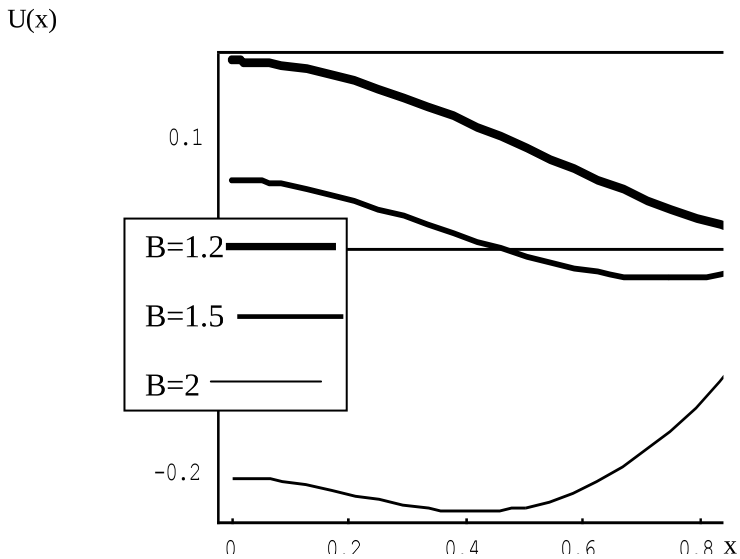

Figure 11. Temperature distribution when B= 1.2, 1.5, 2.

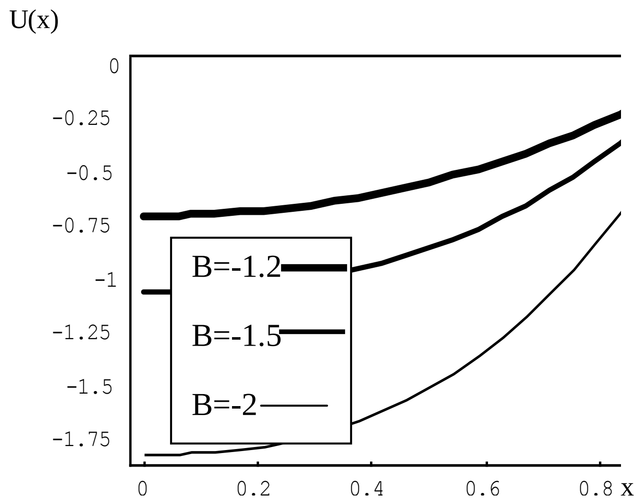

Figure 12. Temperature distribution for B= -1.2, -1.5, -2.

VI. CONCLUSION

This work employs HAM to solve the nonlinear steady state heat conduction problem with temperature dependent thermal conductivity. The results show that B has a very strong influence on the temperature distribution in the bar and cannot always be ignored without error. For some values of B, the temperature distribution in the bar is an increasing function of B while for other values of B it a decreasing function. The results are display in figures. From the analytic expression in equation (21), other heat transfer quantities can be obtained easily.

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