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Some Moonshine connections between Fischer-Griess Monster group (M) and Number theory

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ABSTRACT

This paper investigates moonshine connections between the McKay –Thompson series of some conjugacy classes for the monster group and number theory. We explore certain natural consequences of the Mckay-Thompson series of class 1A, 2A and 3A for the monster group in Ramanujan-type Pi formulas. Numerical results show that, the quadratic prime-generating polynomials are connected to integer values of exactly 43 Mc Kay- Thompson series of the conjugacy classes for the monster group. Furthermore, the transcendental number pi is approximated from monster group.

Keywords: *Fischer-Griess monster group, Monstrous moonshine, Mc Kay- Thompson series, prime-generating polynomial, Fundamental discriminant, Ramanujan-type pi formulas.*

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I. INTRODUCTION

The Fischer-Griess monster group M is the largest and the most popular among the twenty six sporadic finite simple groups of order

$$2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 9^{7}\cdot 11^{2}\cdot 13^{3}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71$$

$$=8080,17424,79451,28758,86459,90496,17107,57005,75436,80000,00000\approx 8\times 10^{53}.$$

In late 1973 Bernd Fischer and Robert Griess independently produced evidence for its existence. Conway and Norton [6] proposed to call this group the *Monster* and conjectured that it had a representation of dimension $196,883=47\cdot 59\cdot 71$.

In a remarkable work, Fischer et. al [8] computed the entire character table of M in 1974 under this assumption. It has 194 conjugacy classes and irreducible characters. The Monster has not yet been proved to exist, but Thompson [14] has proved its uniqueness on similar assumptions. In 1982 Griess [9] constructed M as the automorphism group of his 196884-dimensional algebra thus proving existence. The monster contains all but six of the other sporadic groups as subquotients though their discoveries were largely independent of it.Although the monster group was discovered within the context of finite simple groups, but hints later began to emerge that it might be strongly related to other branches of mathematics. One of these is the theory of modular functions and modular forms.

The elliptic modular function has Fourier series expansion as:

$$j(\tau)=\frac{1}{q}+744+196884q+21493760q^{2}+864299970q^{3}+20245856256q^{4}+333202640600q^{5}+$$
$$+4252023300096q^{6}+\ldots;\ \text{where}\ q=e^{2\pi i\tau},\quad \tau\in\mathrm{H}.$$

In 1978, Mc Kay [17] noticed that the coefficient of q (196884) in the j-function is $196883+1$ and

Thompson [15] found that the later coefficients are linear combinations of the representations of M as given in [1, 2] as follows:

$$1 = 1 \qquad (1.1a)$$

$$196884 = 196883 + 1 \qquad (1.1b)$$

$$21493760 = 21296876 + 196883 + 1 \qquad (1.1c)$$

$$864299970 = 842609326 + 21296876 + 2.196883 + 2.1 \qquad (1.1d)$$

The numbers on the left sides of (1.1) are the first few coefficients of the j-function; where as the numbers on the right are the dimensions of the smallest irreducible representations of the Fischer-Griess monster group M.

Based on these observations, Mc Kay and Thompson [17] found further numerology suggesting that, the explanation for (1.1) should lie in the existence of a natural infinite dimensional graded M-module V_n (later called head representation of M) for the monster $V = \bigoplus_{n \geq 0} V_n$. The dimension of V_n is equal to the coefficient c_n of the elliptic modular function.

II. THE MONSTROUS MOONSHINE

It is vital to mention the work of Conway and Norton [6], which marked as the starting point in the theory of moonshine, proposing a completely unexpected relationship between finite simple groups and modular functions, which relates the monster to the theory of modular forms. Conway and Norton conjectured in this paper that there is a close connection between conjugacy classes of M and action of certain subgroups of $SL_2(\mathbb{R})$ on the upper half plane H. This conjecture implies that extensive information on the representations of the monster is contained in the classical picture describing the action of $SL_2(\mathbb{R})$ on H. Monstrous moonshine is the collection of questions (and few answers) that these observations had directly inspired.

With the emergence of [6], many researchers had presented more results on connections between modular forms and monster group, and most of the other finite simple sporadic groups have been discovered; they are collectively referred to as *Moonshine*. Significant progress was made in the 1990's, and Borcherds won a Field's medal in 1998 for his work in proving Conway and Norton's original conjectures. The proof opened up connections between number theory and representation theory with mathematical physics. For survey see [1, 2, 7].

III. THE MC KAY-THOMPSON SERIES

The central structure in the attempt to understand (1.1) is an infinite-dimensional $\mathbb{Z}$-graded module for the monster .

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus \ldots$$

Let ρ_d denote the d-dimensional irreducible representation of M ordered by dimension. The First subspaces will be:

$V_0 = \rho_1,\ V_1 = \{0\},\ V_2 = \rho_1 \oplus \rho_{196883},\ V_3 = \rho_{21296876} \oplus \rho_{196883} \oplus \rho_1$ and so on

The J-function is essentially its graded dimension

$$J(\tau) = j(\tau) - 744 = \dim(V_0) q^{-1} + \sum_{n=1}^{\infty} \dim(V_n)\, q^n$$

From representation theory for finite groups, a dimension can be replaced with a character. This gives the graded traces

$$T_g(\tau) = \sum_{n \in \mathbb{Z}} \mathrm{tr}\,(g|V_n)\, q^n$$

Generally, Thompson [16] further suggested that, we consider the series (known as the Mc Kay-Thompson series for this module V)

$$T_g(\tau) = q^{-1} \sum_{n=1}^{\infty} \mathrm{ch}_{Vn}(g)\, q^n$$

for each element $g \in M$, where ch_{V_n} are characters.

For example, the smallest non-trivial representation of M is given by almost 10^{54} Complex matrices, each of size 196883 x 196883, while the corresponding character is completely specified by 194 integers (194 being the number of conjugacy classes in M). Taking $g = 1$, we have $T_1(\tau) = J(\tau)$

We write c_n to be the coefficient of q^n in Mc Kay-Thompson series T_g, that is

$$T_g(\tau) = q^{-1} + \sum_{n=1}^{\infty} c_n q^n,\ q = e^{2\pi i\tau}.$$

Moreover, there are 8×10^{53} elements in the monster, so we expect about 8×10^{53} different Mc Kay-Thompson series T_g. However, a character evaluated at g and any of its conjugate elements hgh^{-1} have identical character values and hence have identical McKay-Thompson series $T_g = T_{hgh^{-1}}$. This implies that $T_g = T_h$ whenever the cyclic subgroups $\langle g\rangle$ and $\langle h\rangle$ are equal. Hence there can be at most 194 T_g, one for each conjugacy class. In fact there are 172 distinct McKay-Thompson series T_g and the first eleven coefficients of each are given in [6, Table 4].

Influenced by Ogg's observation, Thompson, Conway and Norton conjectured that for each element g of M, the McKay-Thompson series T_g is the hauptmodul $J_{G_g}(\tau)$ for a genus zero group $G_g \subseteq \Gamma$ of moonshine type. So for each n the coefficient $g \mapsto c_n$ defines a character $ch_{V_n}(g)$ of M. They explicitly identify each of the groups G_g; these groups each contain subgroup $\Gamma_0(N)$ as a normal subgroup, for some N dividing $\circ(g)\cdot(24,\circ(g))$. G_g corresponding to a McKay-Thompson series $T_g(\tau)$ is specified by giving the positive integer N and a subset of Hall divisors of n/h, n arises as the order of g. Then n divides N and the quotient $h = N/n$ divides 24. In fact h^2 divides N.

The full correspondence can be found in [6, Table 2]. The first 50 coefficients c_n of each T_g are given in [10].

IV. PRIME GENERATING POLYNOMIALS AND MONSTER GROUP

In this section, we investigate monster's relationship to quadratic prime-generating polynomials through its conjugacy classes. Piezas [12] showed that for any τ in the quadratic field $\mathbb{Q}([\sqrt{-d(n)}])$. $j(\tau)$ is an algebraic integer . He also showed how prime generating polynomials are connected to integer values of some moonshine functions for small order p. Without the constant terms these moonshine functions are Mckay–Thompson series for M.

Here we consider the McKay-Thompson series of class 1A and some conjugacy classes for monster group and establish relationship to quadratic prime-generating polynomials; by showing that the value $T_p(\tau)$ for an appropriate τ is an algebraic number.

I. $T_{1A}(\tau)$

The Mckay-Thompson series for the monster class 1A is defined by

$$T_{1A}(\tau) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \ldots \text{where } q = e^{2\pi i\tau}$$

This series detects class number **h(-d)=1** of negative fundamental discriminants.

Two forms of τ are given as:

Case 1: $\tau = (1+\sqrt{-d})/2$ (Associated with odd discriminant d).

Case 2: $\tau = \sqrt{-m}$ (Associated with even discriminant $d = 4m$).

For case 1, $q = -e^{-\pi\sqrt{d}}$ and $T_{1A}(\tau)$ is negative.

For case 2, $q = e^{-2\pi\sqrt{m}}$ and $\mathrm{T}_{1A}(\tau)$ is positive.

Table 1 shows their associated quadratic prime-generating polynomial $p(k) = ak^2 + bk + c$, its discriminant d, and $\mathrm{T}_{1A}(\tau)$ using a root τ of $p(k) = 0$, which is given by the quadratic formula

$$k = \tau = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

Table 1. Prime-generating Polynomial for Discriminants with h(-d) = 1 and $\mathbf{T_{1A}(\tau)}$

$\mathbf{p(k) = ak^2 + bk + c}$	$\mathbf{d = b^2 - 4ac}$	$\mathbf{T_{1A}(\tau)}$
$k^2 + 1$	-4	$2^3 \cdot 3 \cdot 41$
$k^2 + 2$	-8	$2^3 \cdot 907$
$k^2 - k + 1$	-3	-751
$k^2 - k + 2$	-7	$-3 \cdot 1373$
$k^2 - k + 3$	-11	$-2^3 \cdot 59 \cdot 71$
$k^2 - k + 5$	-19	$-2^3 \cdot 3 \cdot 5 \cdot 47 \cdot 157$
$k^2 - k + 11$	-43	$-2^3 \cdot 3 \cdot 36864031$
$k^2 - k + 17$	-67	$-2^3 \cdot 3^1 \cdot 6133248031$
$k^2 - k + 41$	-163	$-2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3$

II. $\mathbf{T_P(\tau)}$

The Mckay-Thompson series for the monster is of the form:

$$\mathrm{T_P}(\tau) = \frac{1}{q} + c_1 q + c_2 q^2 + c_3 q^3 + \ldots \qquad \text{where } q = e^{2\pi i\tau}.$$

Two forms of τ:

$$\text{Case 1: } \tau = (1 + \sqrt{-r})/2;\ r = \frac{4c}{a} - 1.$$

$$\text{Case 2: } \tau = \sqrt{-s}\ ;\ s = \frac{c}{a}.$$

For case 1, $q = -e^{-\pi\sqrt{r}}$ and $\mathrm{T_p}(\tau)$ is negative.

For case 2, $q = e^{-2\pi\sqrt{s}}$ and $\mathrm{T_P}(\tau)$ is positive.

A. p as prime divisor of $|\mathrm{M}|$

$\mathrm{p} \in \{2, 3, 5, 7, 9, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$

There is Mckay-Thompson series $\mathrm{T_P}(\tau)$ for each of the 15 prime divisors of $|\mathrm{M}|$.

Table 2 shows their associated quadratic prime-generating polynomials $p(k) = ak^2 + bk + c$, and value $\mathrm{T_p}(\tau)$ using the root τ of $p(k) = 0$

Table 2. Prime-generating Polynomial for Discriminants with h(-d)=2n and $T_{pA}(\tau)$

p	$p(k) = ak^2 + bk + c$	$d = b^2 - 4ac$	h(-d)	$T_{pA}(\tau)$
2	$2k^2-2k+3$	$-20=-2\cdot 10$	2	$-2^3\cdot 3\cdot 47$
2	$2k^2-2k+7$	$-52=-2\cdot 26$	2	$-2^3\cdot 7\cdot 1483$
2	$2k^2-2k+19$	$-148=-2\cdot 74$	2	$-2^3\cdot 56$
2	$2k^2+3$	$-24=-2\cdot 12$	2	$2^3\cdot 5^2\cdot 11$
2	$2k^2+5$	$-40=-2\cdot 20$	2	$2^3\cdot 2579$
2	$2k^2+11$	$-88=-2\cdot 44$	2	$2^3\cdot 313619$
2	$2k^2+29$	$-232=-2\cdot 116$	2	$-2^3\cdot 3073907219$
3	$3k^2-3k+2$	$-15=-3\cdot 5$	2	$-3\cdot 23$
3	$3k^2-3k+5$	$-51=-3\cdot 17$	2	$-2\cdot 3\cdot 5\cdot 59$
3	$3k^2-3k+11$	$-123=-3\cdot 41$	2	$-2\cdot 3\cdot 18439$
3	$3k^2-3k+23$	$-267=-3\cdot 89$	2	$-2^3\cdot 5\cdot 233\cdot 2897$
3	$3k^2+2$	$-24=-3\cdot 8$	2	$2\cdot 3\cdot 29$
5	$5k^2-5k+2$	$-15=-5\cdot 3$	2	$-2^2\cdot 5$
5	$5k^2-5k+3$	$-35=-5\cdot 7$	2	$-5\cdot 11$
5	$5k^2-5k+7$	$-115=-5\cdot 23$	2	$-3^2\cdot 5\cdot 19$
5	$5k^2-5k+13$	$-235=-5\cdot 47$	2	$-5\cdot 7\cdot 1493$
5	$5k^2+2$	$-40=-5\cdot 8$	2	$5\cdot 11$
7	$7k^2-7k+3$	$-35=-7\cdot 5$	2	$-3\cdot 5$
7	$7k^2-7k+5$	$-91=-7\cdot 13$	2	-73
7	$7k^2-7k+17$	$-427=7\cdot 61$	2	$-2^5\cdot 3^2\cdot 37$
11	$11k^2-11k+7$	$-187=-11\cdot 17$	2	$-2\cdot 5^2$
13	$13k^2-13k+5$	$-91=-13\cdot 7$	2	-11
13	$13k^2-13k+11$	$-403=-13\cdot 31$	2	-2^7
17	$17k^2-17k+5$	$-51=-17\cdot 3$	2	-5
19	$19k^2-19k+29$	$-1843=-19\cdot 97$	6	$-\sqrt[3]{1771561000}$
23	$23k^2-23k+7$	$-115=-23\cdot 5$	2	-5
29	$29k^2+2$	$-232=-29\cdot 8$	2	5
31	$31k^2-31k+11$	$-403=-31\cdot 13$	2	-3^2
41	$41k^2-41k+11$	$-123=-41\cdot 3$	2	-3
47	$47k^2-47k+13$	$-235=-47\cdot 5$	2	-5
59	$59k^2-59k+19$	$-1003=-59\cdot 17$	4	$-(3+2\sqrt{2})$
71	$71k^2+2$	$-568=-71\cdot 8$	4	$\sqrt{17}$

B. **p as product of two distinct prime divisors of $|M|$**

$2\cdot\{3,5,7,11,13,17,19,23,31,47\}=\{6,10,14,22,26,34,38,46,62,94\}$,

$3\cdot\{5,7,11,13,17,19,23,29,31\}=\{15,21,33,39,51,69,87,93\}$,

$5\cdot\{7,11,19\}=\{35,55,95\}$, $7\cdot 17=119$

Therefore $p\in\{6,10,14,15,21,22,26,33,34,35,38,39,46,51,55,57,62,69,87,93,94,95,119\}$

There are 23 such orders, though $p=57$ and 93 are to be excluded. For the relevant McKay Thompson series in this family, these two are the only ones where the powers of q are not consecutive but are in the progression $3m+2$. So what remains are 21 series.
In addition, there is Mckay-Thompson series $T_p(\tau)$ for each p. Hence, we present them and their associated $p(k)$ and $T_{pA}(\tau)$ in the following table:

Table 3. Prime-generating Polynomials for Discriminants with h(-d) = 4, 8. 12 and $T_{pA}(\tau)$

p	$p(k)=ak^2+bk+c$	$d=b^2-4ac$	h(d)	$T_p(\tau)$
6	$6k^2-6k+5$	$-84=-6\cdot 14$	4	$-2\cdot 61$
6	$6k^2-6k+7$	$-132=6\cdot 22$	4	$-2\cdot 5\cdot 41$
6	$6k^2-6k+11$	$-228=6\cdot 38$	4	$-2\cdot 23\cdot 59$
6	$6k^2-6k+17$	$-372=-6\cdot 62$	4	$-2\cdot 12157$
6	$6k^2-6k+31$	$-708=-6\cdot 118$	4	$-2\cdot 3\cdot 401\cdot 467$
6	$6k^2+5$	$-120=-6\cdot 20$	4	$2\cdot 5\cdot 31$
6	$6k^2+7$	$-168=-6\cdot 28$	4	$2\cdot 443$
6	$6k^2+13$	$-312=-6\cdot 52$	4	$2\cdot 5\cdot 1039$
6	$6k^2+17$	$-408=-6\cdot 68$	4	$2\cdot 5\cdot 3919$
10	$10k^2-10k+11$	$-340=-10\cdot 34$	4	$-2^3\cdot 41$
10	$10k^2+3$	$-120=-10\cdot 12$	4	2^5
10	$10k^2+7$	$-280=-10\cdot 28$	4	$2^6\cdot 3$
10	$10k^2+13$	$-520=-10\cdot 52$	4	$2^2\cdot 17\cdot 19$
10	$10k^2+19$	$-760=-10\cdot 76$	4	$2^2\cdot 3\cdot 13\cdot 37$
14	$14k^2-14k+5$	$-84=-14\cdot 6$	4	$-3\cdot 59$
14	$14k^2-14k+13$	$-532=-14\cdot 38$	4	$-5\cdot 7$
14	$14k^2+3$	$-168=-28\cdot 6$	4	19
14	$14k^2+5$	$-280=-14\cdot 20$	4	43
15	$15k^2-15k+7$	$-195=-15\cdot 13$	4	-19
15	$15k^2+2$	$-120=-15\cdot 8$	4	11
21	$21k^2-21k+11$	$-483=-21\cdot 23$	4	-3^3
22	$22k^2-22k+7$	$-132=-22\cdot 6$	4	-11
22	$22k^2-22k+17$	$-1012=-22\cdot 46$	4	$-11\cdot 3^2$
26	$26k^2+3$	$-312=-26\cdot 12$	4	3^2
26	$26k^2+5$	$-520=-26\cdot 20$	4	4^2
33	$33k^2-33k+17$	$-1155=-33\cdot 35$	8	$-(20+2\sqrt{5})$
34	$34k^2+7$	$-952=-34\cdot 28$	8	$21-\sqrt{10}$
35	$35k^2+2$	$-280=-35\cdot 8$	4	5
38	$38k^2-38k+13$	$-532=-38\cdot 14$	4	-7
39	$39k^2+2$	$-312=-39\cdot 8$	8	$3\sqrt{11}-5$

46	$46k^2-46k+17$	$-1012=-46\cdot 22$	4	$-2^2\cdot 7$
51	$51k^2+2$	$-408=-51\cdot 8$	4	2^2
55	$55k^2-55k+17$	$-715=-55\cdot 13$	4	-5
62	$62k^2+5$	$-1240=-62\cdot 20$	8	$1+2\sqrt{6}$
69	$69k^2-69k+19$	$-483=-69\cdot 7$	4	-3
87	$87k^2+2$	$-696=-87\cdot 8$	12	$\sqrt[3]{18}$
94	$94k^2+7$	$-2632=-94\cdot 28$	8	$5+\sqrt{5}$
95	$95k^2+2$	$-760=-95\cdot 8$	4	3
119	$119k^2-119k+43$	$-6307=-119\cdot 53$	8	$-(4+\sqrt{17})$

C. p as product of three distinct prime divisors of $|\mathrm{M}|$.

$2\cdot 3\cdot\{5,7,11,13\}=\{30,42,66,78\},\ 2\cdot 5\cdot\{7,11\}=\{70,110\}, 3\cdot 5\cdot 7=105.$

Therefore $\mathrm{p}\in\{30, 42, 66, 70, 78, 105, 110\}$

Table 4 gives their associated quadratic prime-generating polynomial $p(k)$ and $\mathrm{T}_{\mathrm{pA}}(\tau)$.

Table 4. Prime-generating Polynomials for Discriminants with h(-d) = 8 and $\mathbf{T_p(\tau)}$

p	$p(k)=ak^2+bk+c$	$d=b^2-4ac$	$\mathrm{T}_{\mathrm{p}}(\tau)$
30	$30k^2+7$	$-840=-30\cdot 28$	$3\cdot 7$
42	$42k^2+11$	$-1848=-42\cdot 44$	5^2
66	$66k^2+7$	$-1848=-66\cdot 28$	2^3
70	$70k^2-70k+23$	$-1540=-70\cdot 22$	$-2\cdot 3$
78	$78k^2-78k+23$	$-1092=-78\cdot 14$	-2^2
105	$105k^2-105k+31$	$-1995=-105\cdot 19$	-2^2
110	$110k^2+3$	$-1320=-110\cdot 12$	3

Discussion/Remarks

Based on the above presentations, we have the following observations:

1. For p a prime divisor of $|\mathrm{M}|$, $\mathrm{T}_{\mathrm{p}}(\tau)$ is an algebraic number of degree one-half the h(-d). There are 15 such series.

2. For p a product of two distinct prime divisors of $|\mathrm{M}|$ (except p=57 and p=93), the $\mathrm{T}_{\mathrm{p}}(\tau)$ of the appropriate conjugscy class is an algebraic number of degree one-fourth the h(-d). There are 21 such series.

3. For p a product of three distinct prime divisors of $|\mathrm{M}|$, the $\mathrm{T}_{\mathrm{p}}(\tau)$ of the appropriate conjugacy class is an algebraic number of degree one-eighth the h(-d). There are 7 such series.

Thus, there is a total of 15 +21 + 7 = 43 such series, which is equivalent to the least number of conjugacy classes of the maximal subgroups of the monster group.

Note: All the series are from T_{pA} except $\mathrm{T}_{30\mathrm{B}}$, $\mathrm{T}_{33\mathrm{B}}$, $\mathrm{T}_{46\mathrm{C}}$.

1. The Monster group in Ramanujan-type pi formulas

The first series representations for $\frac{1}{\pi}$ were found by an Indian mathematical genius Srinivasa Ramanujan [13], he discusses certain methods to derive exact and approximate evaluations but first proved by the Borweins [4]. The work of the Borweins [3], [4], and the Chudnovskys [5] extends Ramanujan's work. Complete list of 36 such pi formulas is given in [12].

In this section we explore certain natural consequences of the Mckay-Thompson series of class 1A, 2A and 3A for the monster group in 3 kinds of Ramanujan-type pi formulas; by considering an example of each kind.

Piezas[12] in his approach considered the polynomials $p(k)=k^2-k+41$, $p(k)=2k^2-2k+19$, $p(k)=3k^2-3k+23$, $p(k)=4k^2-4k+5$ and related moonshine functions, $j(\tau),\gamma_{pA}$, for $p=2,3,4$. He pointed out that the values of $j(\tau),\gamma_{pA}$, for $p=2,3,4$ appeared to be in 4 different kinds of Ramanujan-type pi formula. In [14] , he remarked that, the Variable C in the general form of these pi formulas is either the value of $j(\tau)$ or γ_{pA}, $p=2,3,4$ for an appropriate τ and conveniently expressed A interms of C.

In our own contribution we consider the following prime-generating polynomials of form

$p(k)=ak^2-ak+c$,

$p(k)=k^2-k+11$

$p(k)=2k^2-2k+7$

$p(k)=3k^2-3k+11$

And the following related series of the monster group,

$$T_{1A}(\tau)=\frac{1}{q}+196884q+21493760q^2+864299970q^3+...=j(\tau)-744$$

$$T_{2A}(\tau)=\frac{1}{q}+4372q+96256q^2+1240002q^3+...=\gamma_{2A}-104$$

$$T_{3A}(\tau)=\frac{1}{q}+783q+8672q^2+65367q^3+...=\gamma_{3A}-42$$

$$where\ q=e^{2\pi i\tau}.$$

Solving the polynomials at the point $p(k)=0$, then,

$$k=\tau=(1+\sqrt{-43})/2$$

$$k=\tau=(2+\sqrt{-52})/4=(1+\sqrt{-13})/2$$

$$k=\tau=(3+\sqrt{-123})/6=(1+\sqrt{-41/3})/2$$

Where, consistent with the requirements for modular forms, we have chosen the root τ that is on the upper half complex plane.

Plugging τ into q, we got the very small real numbers,

$$q=-\exp(-\pi\sqrt{43})$$

$$q=-\exp(-\pi/2\sqrt{52})=-\exp(-\pi\sqrt{13})$$

$$q=-\exp(-\pi/3\sqrt{123})=-\exp(-\pi\sqrt{41/3})$$

Which when substituted into the appropriate series, yields the integers,

$$T_{1A}((1+\sqrt{-43})/2)\approx-884736744=\mathbf{-960^3}-744$$

$$T_{2A}((1+\sqrt{-13})/2)\approx-83048=\mathbf{-288^2}-104$$

$$T_{3A}((1+\sqrt{-41/3})/2)\approx-1106324=\mathbf{-48^3}-42$$

These integers written in bold appeared in three of four different kinds of pi formulas.

The first integer $\mathbf{-960^3}$ appears in the Chudnovsky brothers' Pi formula,

$$\frac{1}{\pi}=12\sum_{k=0}^{\infty}\frac{(-1)^k(6k)!}{(k!)^3(3k)!}\left(\frac{16254k+789}{(960^3)^{k+1/2}}\right).$$

This can be written as:

$$\frac{1}{\pi}=12\sum_{k=0}^{\infty}\frac{(-1)^k(6k)!}{(k!)^3(3k)!}\left(\frac{43(378)k+789}{(960^3)^{k+1/2}}\right)$$

Where one can see $|d|=43$,

The second integer, $\mathbf{-288^2}$, appears in a Pi formula found by Ramanujan [9] in 1912 (sixty years before the monster group was even discovered)

$$\frac{1}{\pi}=4\sum_{k=0}^{\infty}\frac{(-1)^k(4k)!}{(k!)^4}\left(\frac{260k+23}{(288^2)^{k+1/2}}\right)$$

This can be written as:

$$\frac{1}{\pi}=4\sum_{k=0}^{\infty}\frac{(-1)^k(4k)!}{(k!)^4}\left(\frac{52(5)k+23}{(288^2)^{k+1/2}}\right)$$

where $|d|=52$ also appears,

The third, $\mathbf{-48^3}$, appears in another kind of Pi formula found by Chudnovsky brothers [5].

$$\frac{1}{\pi}=2\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!(3k)!}{(k!)^5}\left(\frac{615k+53}{(48^3)^{k+1/2}}\right)$$

This can be written as:

$$\frac{1}{\pi}=2\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!(3k)!}{(k!)^5}\left(\frac{123(5)k+53}{(48^3)^{k+1/2}}\right)$$

where one can see $|d|=123$

The general form of these Pi formulas is:

$\frac{1}{\pi}=\sum_{k=1}^{\infty}h_p(\mathrm{A}k+\mathrm{B})/\mathrm{C}^k$ for $\mathrm{p}=1,2,3$.

where A, B and C are algebraic numbers and h_p is the factorial quotients defined as:

$h_1=(6k)!/((3k)!k!^3),\ h_2=(4k)!/(k!^4),\ h_3=(2k)!(3k)!/(k!^5),$

Or equivalently the h_p can be defined as Pochhammer symbol products,

$$h_1=1728^k(1/2)_k(1/6)_k(5/6)_k/(k!^3),\ h_2=256^k(1/2)_k(1/4)_k(3/4)_k/(k!^3),\ h_3=108^k(1/2)_k(1/3)_k(2/3)_k/(k$$

Where $(a)_k=(a)(a+1)(a+2)\cdots(a+k-1)$, with limiting ratio $L_1=1728,\ L_2=256,\ L_3=108$ between successive terms as $k\to\infty$.

Piezas [13] pointed out that the variable C is the value of the moonshine function $\gamma_{\mathrm{pA}}(\tau)$ for an appropriate τ, where $\mathrm{p}=1,\ 2,\ 3$ in the three kinds of pi formulas and try to expressed A in terms of C.

To this end, we conveniently expressed A in terms of d as follows:

- For h_1, and $\mathrm{T_{1A}}(\tau)$: $\mathrm{A}_1=\sqrt{d}$;
- For h_2 and $\mathrm{T_{2A}}(\tau)$: $\mathrm{A}_2=(1/2)\sqrt{d}$
- For h_3 and $\mathrm{T_{3A}}(\tau)$ $\mathrm{A}_3=(1/3)\sqrt{d}$

Hence for h_p and $T_{pA}(\tau)$ $A_p = (1/p)\sqrt{d}$, for $p = 1, 2, 3.$

The discriminant d depends on the form of $\tau = (a + \sqrt{-d})/2a$ for $a = 1, 2, 3$

2. An Approximation of pi from Monster group

Monster group gives more impressive approximation to $e^{\pi\sqrt{d}}$ $(d \in \square)$.

The famous Ramanujan's pi approximation made use of the near-integer result of $e^{\pi\sqrt{163}}$.
163 being the largest discriminant with class number one.
It is vital to mention that, many researchers have presented numerous approximations of pi from monster Group. In the work of Plouffe, Borwein, and Barley (2003), the pi approximation is given as; $\pi \approx \ln(5280)\cdot\sqrt{9/67} = 3.14159265$ (9d) while Warda (2004) presented a new approach as

$$\pi \approx \frac{\ln(640320^3 + 744)^2 - 2\cdot 196884}{2\cdot\sqrt{163}} 3.14159265335897932384626433832795028841971693993 \text{ (46d)}$$

Moreover, Jim (2009) introduces another way using the Ramanujan constant as:

$$\pi \approx \frac{\ln(640320^3 + 744)}{2\cdot\sqrt{163}} = 3.141592653358979323846264338327 \text{ (30d)}$$

In his work, the pi approximation has been improved to 75 decimal digits. In addition, from pi approximation contest centre it has been presented that pi approximation can be obtain via:

(i) $\pi \approx \dfrac{\ln(960^3 + 744)}{\sqrt{43}} = 3.1415926535898$ (13d)

(ii) $\dfrac{3\ln(640320)}{\sqrt{163}} = 3.141592653589793$ (16d)

Based on the above contributions by many researchers we extent this approach and produced a good and stable approximation to pi. To this end**,** we will here make use of the relation $e^{\pi\sqrt{67}}$, 67 being the second largest discriminant with class number one.

$$e^{\pi\sqrt{67}} = 147197952743.99999866245424450683\ldots$$

$$j((1+\sqrt{-67})/2) \approx 147197952744 = 5280^3 + 744$$

We will here denote the near-integer by A and its estimate A_{e_1}

That is, $A = e^{\pi\sqrt{67}} \approx A_e = 147197952744 = 5280^3 + 744$

Then $e^{\pi\sqrt{67}} \approx 5280^3 + 744$

Taking the natural logarithm of both sides of the resulting equation and then dividing both sides by $\sqrt{67}$ we have,

$$\pi \approx \frac{\ln(5280^3 + 744)}{\sqrt{67}} \approx 3.141592653589793239\mathbf{57}\ldots \quad \text{(18d)}.$$

The accuracy of the approximation is remarkable, 18 places of decimal are accurately reproduced (the last three digits that are incorrect are typed in bold).
What if we could make a better estimate of A:
Our estimate A_{e_1} is slightly larger than A by an amount:

$$A_{e_1} - A = 1.3375457755\ldots\times 10^{-6}$$

When we multiply this slight difference by 5280^3

$$5280^3\cdot(A_{e_1} - A) = 196883.998859851776\ldots$$

Remember that 196883 is the dimension of the smallest irreducible representation of the monster group.

The above result is almost exactly this number plus one. we can now write our second estimate A_{e_2} of A as:

$$A_{e_2} = 5280^3 + 744 - \frac{196884}{5280^3}$$

and we construct a new π approximation as:

$$\pi \approx \frac{\ln(A_{e_2})}{\sqrt{67}} = 3.14159265358979323846264433\mathbf{768}... \quad (26d)$$

What if we continue this kind of analysis one step further? In the last step, we correct A_e by subtracting $\frac{196884}{5280^3}$ and called this improved estimate A_{e_2} .

We can assume that we are still in error by some small amount and we will find that our estimate is just a pinch too low:

$$A_{e_2} - A = -7.74567993989481592787377 9...\times 10^{-15}$$

One way to handle this is multiply the difference by 5280^6 to obtain the following result:

$$5280^6(A_{e_2} - A) = -167827483.549237248...$$

Since we have another near–integer, we can round this number off and use it to construct our third estimate A_{e_3}

$$A_{e_3} = 5280^3 + 744 - \frac{196884}{5280^3} + \frac{167827484}{5280^6}$$

Again we construct a new π approximation formula as:

$$\pi \approx \frac{\ln(A_{e_3})}{\sqrt{67}} = 3.14159265358979323846264338327\mathbf{380}... \quad (30d)$$

Continue on the fourth step;

$$A_{e_3} - A = -6.8299999999999999167...\times 10^{-18}$$

$$5280^9(A_{e_3} - A) = -21783417061653302.26565873664\ldots$$

the next approximation would be A_{e_4} and would be equal to:

$$A_{e_4} = 5280^3 + 744 - \frac{196884}{5280^3} + \frac{167827484}{5280^6} + \frac{21783417061653302}{5280^9}$$

$$\pi \approx \frac{\ln(A_{e_4})}{\sqrt{67}} = 3.14159265358979323846264338327 95...$$

We cannot continue with further corrections to see what the results may be,

as the estimate A_{e_3} is equal to the value of A.

We hope that our approach in finding the approximate value of pi from monster group is better and accurate compared to some other related work.

V. CONCLUSION

1. There are many amazing aspects on the McKay-Thompson series of some conjugacy classes for monster group in the theory of moonshine.
2. Remarkable relationship exists between two seemingly different mathematical objects π and e on one hand, and monster group on the other.

Finally, we can say that, the most exciting prospects for the future of moonshine are in the direction of number theory

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