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A Stability Analysis for a Two-Sex Mathematical Model of Hiv

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ABSTRACT
We Analyze A Two-Sex Model Of HIV Epidemics For Stability Of The Infection-Free Equilibrium And Infection Equilibrium. We Formulate Theorems Using The Basic Reproduction Number R_o And Determine The Criteria For Stability Of The Two Equilibrium Points. Our Results Showed That The Infection Is Cleared From The Population If $R_o < 1$ While The Infection Is Persistent And Could Lead To Full Blown AIDS If $R_o > 1$.

KEYWORDS: *HIV/AIDS, Equilibrium Points, Basic Reproduction Number, Stability.*

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I. INTRODUCTION

HIV/AIDS is a deadly disease of greatest public concern. According to the estimates of the Joint United Nations Program on HIV/AIDS (UNAIDS), an estimated 4.1 million people acquired HIV in 2005,bringing the number of people globally living with the virus to approximately 38.6 million [9].This pandemic has claimed more than 2.8 million lives in 2005,bringing the total death toll to 20 million people [9].Current projections suggest that an additional 45 million people will become infected with the virus between 2007 and 2013 unless the world succeeds in mounting a drastically expanded global preventive effort [10]. Despite ongoing efforts to limit its spread it continues to cause epidemic and pandemic infection [1, 6, 9].It has reduced life expectancy, deepened poverty and eroded the capacity of governments to provide essential services. The prevalence of infection has continued to grow due to a very strong non-linear transmission dynamics of the virus and the co-infection of tuberculosis and HIV epidemics [5]. Various mathematical models have been studied to control the spread of the disease [1, 3, 7, 8, 9]. In particular, the two-sex version of a mathematical model of Kimbir [8] studied the effect of condom for the prevention of HIV and demonstrates numerically that HIV infection can be eradicated in finite time if the proportion of infective males using the preventive measure is sufficiently increased. In this paper, we are interested in the mathematical analysis of the equilibrium points of the model that were not previously considered in [8]. We formulate theorems based on the basic reproduction number R_o.

II. MATHEMATICAL FORMULATION AND STABILITY OF EQUILIBRIUM POINTS

The model is [8]

$$y_1' = c_1\beta_2 y_2(1-y_1-w) - y_1 b(1-w) + \alpha y_1(y_1+w) - y_1(\alpha+\rho) \qquad (1)$$
$$y_2' = \rho y_1 - wb(1-w) - \alpha w(1-y_1-w) \qquad (2)$$
$$w' = (1-y_2)(c_2\beta_1 y_1 - \alpha y_2) - by_2(1-w) \qquad (3)$$

Where
1 denotes male
2 denotes female
c_i denotes average sexual contact partners for both sex i $(i = 1,2)$
β_i denotes transmission rates for sex i $(i = 1,2)$
ρ denotes the proportion of male infectives who use preventive measure per unit time
b *denotes* recruitment rate sexually active members of the population(assumed equal for both sexes)
N_i denotes population of i
I_i denotes infected i
y_i denotes I_i/N_i

w denotes population of the male who use condom

Suspending the preventive measure (i.e $\rho = 0$,w=0), we modify the equations in [8] to obtain the translated equations corresponding to the critical points H_o and H*

$$y_1' = -(b-\alpha)y_1 - \frac{c_1\beta_2}{b+\alpha}y_2 \quad (4)$$

$$y_2' = c_2\beta_1 y_1 - (b+\alpha)y_2 \quad (5)$$

and

$$y_1' = \frac{c_1\beta_2\gamma}{b+\alpha-\gamma}y_1 - c_1\beta_2 y_2 \quad (6)$$

$$y_2' = c_2\beta_1 y_1 - \frac{c_2\beta_1\mu}{b+\alpha-\mu}y_2 \quad (7)$$

where H_o and H* are infection free and infection equilibrium points of equations(4)-(7) and are given by

$H_o = (0,0)$ and $H^* = (\frac{c_1\beta_2\gamma}{b+\alpha-\gamma}, \frac{c_2\beta_1\mu}{b+\alpha-\mu})$.

2.1 THE BASIC REPRODUCTION NUMBER

The basic reproduction number R_0 is the number of secondary infections induced by an infected individual introduced into the total susceptible population.Using the formulation of R_o in [3], we define the basic reproduction number of model (1)-(3) as

$$R_0 = c_1\beta_2 c_2\beta_1 / b + \alpha \quad (8)$$

2.2 STABILITY THEOREMS

We shall need the theorems below in the stability analysis of the equilibrium points

Theorem 1 [4]

Let $\frac{dx}{dt}$ = P(x,y), $\frac{dy}{dt}$ = Q (x,y) and X = $\begin{pmatrix} x \\ y \end{pmatrix}$

Let x_1 = $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ be a critical point of the plane autonomous system X_1 = g(x) = $\begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}_1$, where P(x,y) and Q(x,y) have continuous first partial derivatives in a neighbourhood of X_1

(a) If the eigenvalues of A = $g^1(X_1)$ have negative real part then X_1 is an asymptotically stable critical point

(b) If A = $g^1(X_1)$ has an eigenvalue with positive real part, then X_1 is an unstable critical point.

Theorem 2 [2] (DESCARTES' RULE OF SIGNS)

The number of positive zeros (negative zeros) of polynomials with real coefficient is either equal to the number of change in sign of the polynomial or less than this by an even number (By counting down by two's).

2.3 STABILITY OF THE INFECTION FREE EQUILIBRIUM

Theorem 3

The infection free equilibrium is asymptotically stable if $k_1>0$, $k_2>0$ and $R_o<1$.

Proof

The Jacobian matrix of the infection free equilibrium is

$$G = \begin{pmatrix} -b-\alpha & c_1\beta_2 / b+\alpha \\ c_2\beta_1 & -b-\alpha \end{pmatrix} \quad (8)$$

The eigenvalues of the infection free equilibrium is obtained by solving

$$|G - \lambda I| = 0 \quad (9)$$

$$(-b-\alpha-\lambda)(-b-\alpha-\lambda) - c_1\beta_2 c_2\beta_1 / b + \alpha = 0 \quad (10)$$

That is

$$\lambda^2 + k_1\lambda - (k_2 - R_0) = 0 \quad (11)$$

Where

$$k_1 = 2b + 2\alpha \qquad (12)$$

$$k_2 = b^2 + \alpha^2 + 2b\alpha \qquad (13)$$

The solutions of (11) are

$$\lambda_1 = \frac{-k_1 + \sqrt{k_1{}^2 - 4(k_2 - R_0)}}{2} \qquad (14)$$

$$\lambda_2 = \frac{-k_1 - \sqrt{k_1{}^2 - 4(k_2 - R_0)}}{2} \qquad (15)$$

Now, if $k_1 > 0, k_2 > 0$ and $R_0 < 1$, it follows that λ_1 and λ_2 are both less than zero i.e. are both negative.
There are no sign changes and hence by Descartes' rule of signs, all the eigenvalues are negative.
Therefore, H_0 is asymptotically stable.
Remark 1
Using realistic model parameter values [8], $\alpha = 0.1, b = 0.5, c_1 = c_2 = 0.6, \beta_1 = \beta_2 = 0.14$.
Then, $k_1 = 1.2, k_2 = 0.36$ and $R_0 = 0.15$. By the conditions of theorem 3,it shows that the infection free equilibrium is asymptotically stable.

Theorem 4
The infection-free equilibrium is unstable if $k_1 > 0, k_2 > 0$ and $R_0 > 1$
Proof
It is sufficient to show that at least one eigenvalue is positive.
From equation (13)-(14) ,if we let $k_1 > 0, k_2 > 0$ and $R_0 > 1$,it follows then that there is only one sign change which according to Descartes' rule of signs implies there is exactly one positive root and then the infection free equilibrium point H_0 is unstable.

Remark 2
If we take parameter values in model system (1)-(3) as
$\alpha = 0.3, b = 0.24, c_1 = c_2 = 5, \beta_1 = \beta_2 = 0.3$,We calculate
$k_1 = 0.54, k_2 = 0.29, R_0 = 4.1667$. It shows that the infection free equilibrium point is unstable.

2.4 STABILITY OF THE INFECTION EQUILIBRIUM

Theorem 5
The infection equilibrium asymptotically stable if $r_1 < 0, r_2 > 0$ and $R_0 < 1$.

Proof
The Jacobian matrix of infection equilibrium H^* is

$$A = \begin{pmatrix} \frac{c_1\beta_2\gamma}{b+\alpha-\gamma} & c_1\beta_2 \\ c_2\beta_1 & \frac{c_1\beta_2\mu}{b+\alpha-\mu} \end{pmatrix} \qquad (16)$$

The eigenvalues of infection free equilibrium is obtained by solving

$$|A - \lambda I| = 0 \qquad (17)$$

That is,

$$\left(\frac{c_1\beta_2\gamma}{b+\alpha-\gamma} - \lambda\right)\left(\frac{c_2\beta_1\mu}{b+\alpha-\mu} - \lambda\right) - c_1\beta_2c_2\,\beta_1 = 0$$

$$\lambda^2 - r_1\lambda + r_2(1 - R_0) = 0 \qquad (18)$$

Where

$$r_1 = \frac{c_1\beta_2\gamma}{b+\alpha-\gamma} - \frac{c_2\beta_1\mu}{b+\alpha-\mu} \quad (19)$$

$$r_2 = \frac{c_1\beta_2 c_2\beta_1}{(b+\alpha-\mu)(b+\alpha-\gamma)} \quad (20)$$

Solving (18), we obtain the two eigenvalues of the infection free equilibrium as

$$\lambda_1 = \frac{-r_1-\sqrt{r_1{}^2-4r_2(1-R_0)}}{2} \quad (21)$$

$$\lambda_2 = \frac{-r_1+\sqrt{r_1{}^2-4r_2(1-R_0)}}{2} \quad (22)$$

If $r_1 < 0, r_2 > 0$ and $R_0 > 1$, it follows then that λ_1 and λ_2 are both less than zero.There are no variation in sign and hence by Descartes' rule of signs all the eigenvalues are negative or two negative roots.Therefore,by theorem 5,the infection equilibrium point of system (1)-(3) is asymptotically stable.

Theorem 6

The infection equilibrium is unstable if $r_1 > 0, r_2 < 0$ and $R_0 > 1$.

Proof

From equations (21) and (22) of the proof of theorem 5, if we let $r_1 > 0, r_2 < 0$ and $R_0 > 1$, then there is only one sign change in equation (18).Hence, by Descartes' rule of signs, equation (18) has a positive root and then the infection equilibrium point is unstable.

III. DISCUSSION OF RESULTS AND CONCLUSION

We have discussed the stability of equilibrium points that were not previously consideredin [8].The analysis revealed that the infection equilibrium is asymptotically stable if $k_1 > 0, k_2 > 0$ and $R_0 < 1$ and unstable if $k_1 > 0, k_2 > 0$ and $R_0 > 1$.Furthermore,the stability criteria shows that the infection equilibrium is asymptotically stable if $r_1 > 0, r_2 < 0$ and $R_0 < 1$, and unstable if $r_1 > 0, r_2 < 0$ and $R_0 > 1$.A key factor in the analysis is R_0. Virus infection is temporal and can be cleared when $R_0 < 1$ and other measures such as improved HIV vaccines,ant-retroviral drugs and media alert of the disease are taken concurrently to control the spread of the infection.However, the infection persists when $R_0 > 1$ and the long periods of infectiousness could result into an epidemic which could lead to AIDS and a wiping out of the total population if drastic measures are not taken to prevent its spread. Hence, the study of the infection equilibrium state of our model and similar models of HIV has important health implications for disease control and elimination. Mathematicians should continue to develop mathematical models that would provide information for medical practitioners to find optimal drugs for the control of HIV so that a stable infection equilibrium does not exist.

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