The International Journal Of Engineering And Science (IJES)
||Volume|| 3 ||Issue|| 1 ||Pages|| 18-26 || 2014 ||
ISSN (e): 2319 – 1813 ISSN (p): 2319 – 1805

On cycle index and orbit – stabilizer of Symmetric group

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---ABSTRACT---
In this paper explicit formulas are used to find the cycle index of permutation group especially the symmetric groups, the alternating group and the number of orbits of A. Finally we presented the orbits – stabilizer of symmetric group.

Keywords: Cycle index, Orbits, Stabilizer, Permutation group, Symmetric group.

Date of Submission: 20 August 2013 ⟺ Date of Acceptance: 25 January 2014

I. INTRODUCTION

Let A be a permutation group with object set $X = \{1,2,3,\ldots\}$. It is well known that each permutation α on A can be written uniquely as a product of disjoint cycles. Every permutation group has associated with polynomials called the "cycle index".The concept can be traced back to Frobenius [3] as a special case of a formulation in terms of a group characters. Rudvalis and Snapper [3],[5] point out the connection between these generalized characters and theorems of deBruijn [6] and Foullres [3] Redfield [2] who discovered cycle indexes independently devised a clever schemes which enable him to determine the number of classes of certain matrices by forming a special product of cycle indexes.
In this paper, we established the connection between cycle index and orbit – stabilizer of a symmetric group.

II. PRELIMINARIES AND DEFINITION

2.1. The following notations are used:

- S_n denotes the symmetric group.
- $Z(S_n)$ denotes the cycle index of symmetric group.
- $N(A)$ is the number of orbits of group of A
- $Z(A_n)$ denotes the alternating group of A

2.2. Definition

(i) A cycle of length k is a permutation for which there exists an element x in $\{1,2,3,\ldots,n\}$ such that , $x, f(x), f^2(x), \ldots, f^k(x) = x$ are the only elements of f . It is required that $k \geq 2$ since with $k = 1$ the elements x itself would not be moved either.

(ii) The cycle index of a permutation group, G is the average of the cycle index monomial of all the permutation g in G.

(iii) Consider a group G on a set X, the orbit of a point x in X is a set of X to which x can be moved by the element of G.

The orbits of x is defined by $G_x = \{g \in G : g \in G\}$

The associated equivalent relation is defined by saying $X \sim Y$ if and only if there exist $g \in G$ with $g.x = y$

The elements x and y are equivalent if and only if their orbits are the same i.e. $G_x = G_y$

The set G_x js called a subset of x called the orbits of x under$G$.

(iv) For every $x\ in\ X$, we define the stabilizer subgroup of x as the set of all elements in G that fiz x. $G_x = \{g \in G : g.x = x\, where\ x \in G\}$
This is a subgroup of G. The action of $\alpha\ in\ X$ is free if and only if all stabilizer are all trivial.

III. CYCLE INDEX OF SYMMETRIC GROUOP

Let A be a permutation with object set $X = \{1,2,3,\ldots\}$. It is well known that each permutation $\alpha\ in\ A$ can be written uniquely as a product of disjoint cycles and so for each integer $k\ fom\ 1\ to\ n$, we let $J_k(\alpha)$ be the number of cycles of length k in a disjoint cycle decomposition of α.
Then the cycle index of A denoted by $Z(A)$, (Z from the word Zyklenzeiger) is the polynomial in the variables $S_1, S_2, \ldots, S_n$ define by

$$Z(A) = |A^{-1}| \sum_{\alpha \in A} \prod_{k=1}^{n} s_k^{J_k(\alpha)} \qquad \ldots.(2.1)$$

or it is written as $Z(A: s_1, s_2, \ldots, s_n)$.
Consider the symmetric group S_3 on n object for $n = 3$

$$Z(S_3) = \frac{1}{3\,!} \sum_{\alpha \in A} \prod_{k=1}^{n} s_k^{J_k(\alpha)}$$

$$S_3 = \{(1)(2)(3),\ (1)(23),\ (2)(13),\ (12)(3),\ (123),\ (132)$$

For $k = 1$, $S_1 = J_1(\alpha) = 3$ i.e the number of cycles of length k
For $k = 1\ and\ 2$ $S_1 S_2 = J_2(\alpha) = 1$
For $k = 3$, $S_3 = J_3(\alpha) = 1$
This implies that $Z(S_3) = \frac{1}{3}\,!\{s_1^3 + 3s_1 s_2 + 2s_3\}$
Where S_1^3 implies 3 cycle of length 1 occurring together i.e in the same permutation or in one pair (identity).
$3S_1S_2$ implies cycle of length 1 and 2 occurring together in 3 different pairs.
$2S_3$ implies cycle of length 3 occurring in 2 different pairs.
For S_2 we have
$Z(S_2) = \frac{1}{2\,!}\{s_1^2 + s_2\}$ i.e $S_2 = \{(1)(2), (12)\}$
For $k = 1$, $S_1 = J_1(\alpha) = 2$
For $k = 2$, $S_2 = J_2(\alpha) = 1$
s_1^2 implies 2 cycles of length of one occurring together in one pair.
s_2 implies 1 cycle of length 2 occurring once.
Therefore if the length are in one pair, it will be raised to the power of $J_k(\alpha)$ if they are more than one we add them together i.e different pairs and s_1, s_2, s_3 means lenght1, lenght2 and length 3 respectively and so on.
Note that each permutation $\alpha\ of\ n$ object can be associated with the partition of n which has for each k.
Let $h(j)$ be the number of permutation in S_n, whose cycle decomposition will determine the partition (j), so that for each , $J_k = J_k(\alpha)$
Then is easy to see that

$$h(j) = \frac{n!}{\prod k^{j_k} . J_k!} \qquad \ldots (3.2)$$

Example

$$S_3 = \{(1)(2)(3),\ (1)(23),\ (2)(13),\ (12)(3),\ (123),\ (132)$$

For $k = 1$, $h(j) = \frac{3\,!}{1^3.3!} = \frac{6}{6} = 1$
For $k = 2$, $h(j) = \frac{3\,!}{2^1.1!} = \frac{6}{2} = 3$

For $k = 3$, $h(j) = \frac{3\,!}{3^1.1!} = \frac{6}{3} = 2$

which gives $s_1^3 + 3s_1s_2 + 2s_3$

also for $S_2 = 2!$ i.e $S_2 = \{(1)(2), (12)\}$

For $k = 1$, $h(j) = \frac{2\,!}{1^2.2!} = \frac{2}{2} = 1$

For $k = 2$, $h(j) = \frac{2\,!}{2^1.1!} = \frac{2}{2} = 1$

Hence we have $1s_1^2 + 1s_2$

For $k = 1$, $S_1 = 1! = 1$

$h(j) = \frac{2\,!}{1^2.1!} = \frac{1}{1} = 1$

which gives $1s_1$.

Thus the cycle index $Z(S_n)$ takes the form shown in the next theorem.

Theorem 3.1
The cycle index of the symmetric group is given by

$$Z(S_n) = (\tfrac{1}{n\,!}) \sum_j h(j) \prod_{k=1}^{n} s_k^{J_k} \qquad \ldots \quad (3.3)$$

Corollary 3.1
The cycle index of the alternating group is given by

$$Z(A_n) = Z(S_n) + Z(S_n: s_1, -s_2, s_3, -s_4, \ldots) \qquad \ldots..(3.4)$$

To illustrate, we know that

$$Z(S_3) = \frac{1}{3!}\{s_1^3 - 3s_1s_2 + 2s_3\}$$

$$Z(S_3: s_1, -s_2, s_3) = \frac{1}{3!}\{s_1^3 - 3s_1s_2 + 2s_3\}$$

It implies $Z(A_n) = \frac{1}{3!}\{s_1^3 + 2s_3\}$

Also for S_2

$$Z(S_2) = \frac{1}{2!}\{s_1^2 + 2s_2\}$$

$$Z(S_2: s_1 - s_2,) = \frac{1}{2!}\{s_1^2 + s_2\}$$

Which gives $Z(A_2) = \frac{1}{2!}(s_1^2)$

It is often convenient to express $Z(S_n)$ in terms of $Z(S_k)$ with $k < n$. for this purpose we define $Z(S_0) = 1$ and this give a recursive formula, whose inductive proof is straight forward and can be stated as follows.

Theorem 3.2
The cycle index of the symmetric group satisfies the recurrence relation

$$Z(S_n) = \frac{1}{n}\sum_{k=1}^{n} s_k Z(S_{n-k})$$

which is the general formula for the cycle index for S_n

Example

$Z(S_0) = 1$

$Z(S_1) = 1! = 1$

$Z(S_2) = 2! = 2$

$$Z(S_2) = \frac{1}{2}\{s_1 Z(S_{2-1}) + S_2 Z(S_{2-2})\}$$

$$= \frac{1}{2}\{s_1 Z(S_1) + S_2 Z(S_0)\}$$

$$= \frac{1}{2}\{s_1(S_1) + S_2(1)\} = \frac{1}{2}\{S_1(S_1 + S_2)\}$$

$$Z(S_3) = \frac{1}{3}\{s_1 Z(S_{3-1}) + S_2 Z(S_{3-2}) + S_3 Z(S_{3-3})\}$$

$$= \frac{1}{3}\{s_1 Z(S_2) + S_2 Z(S_2) + S_3 Z(S_0)\}$$

$$= \frac{1}{3}\left\{s_1\left(\frac{1}{2}s_1^2 + s_2 s_1\right) + s_2(s_1) + s_3(1)\right\} = \frac{1}{3}\{\frac{1}{2}s_1^3 + \frac{s_1 s_2}{2} + s_1 s_2 + s_3\}$$

$$\frac{1}{6}\{s_1^3 + 3s_1 s_2 + 2s_3\}$$

$$Z(S_4) = \frac{1}{4}\{s_1 Z(S_{4-1}) + s_2 Z(S_{4-2}) + s_3 Z(S_{4-3}) + s_4 Z(S_{4-4})\}$$

$$= \frac{1}{4}\{s_1 Z(S_3) + s_2 Z(S_2) + s_3 Z(S_1) + s_4 Z(S_0)\}$$

$$= \frac{1}{4}\left\{s_1\left(\frac{1}{6}s_1^3 + 3s_1 s_2 + 2s_3\right)\} + s_2\{\frac{1}{2}(s_1^2 + s_2)\right\} + s_3(s_1) + s_4(1)\})$$

$$= \frac{1}{4}\{\frac{S_1^4}{6} + \frac{3S_1^2 S_2}{6} + \frac{S_1^2 S_2}{2} + \frac{S_1^2}{2} + \frac{2S_1 S_3}{6} + s_1 s_3 + s_4\}$$

$$= \frac{1}{4}\{\frac{S_1^4}{6} + \frac{6S_1^2 S_2}{6} + \frac{S_2^2}{2} + \frac{8S_1 S_3}{6} + s_4\}$$

$$\frac{1}{24}\{s_1^4 + 6s_1^2 s_2 + 3s_2^2 + 6s_4\}$$

IV. ORBIT – STABILIZER OF SYMMETRIC GROUP

For each of $x \; in \; X$, let $A(x) = \{\alpha \in A \mid \alpha x = x\}$. Thus $A(x)$ is called the stabilizer of x. Let G be the group of permutation of a set S, for each j in S, let stabilizer $G(j) = stab_G(j)$.Then $stab_G(j) = \{a \in G \mid j\alpha = j\}$.

Theorem 4.1

For any element y of an orbit Y of A, that is the elements in the orbits of y is the index of the stabilizer of y in A.

$|A| = |A(y)||Y|$ …. (4.1)

Proof
To see this, we first express A as a union of right cosets modulo $A(Y)$

$A = \bigcup_{i=1}^{m} \alpha_i A(y)$ … (4.2)

It only remain to observe the natural 1 – 1 corresponding between these cosets and the elements of Y. For each $i = 1$ to m, we associated the coset $\alpha_i A(y)$ with the elements $\alpha_i(y)$ in Y. For $i \neq j$ we have $\alpha_i(y) \neq \alpha_j(y)$ because otherwise $\alpha_j^{-1}\alpha_i$ is an elements of $A(y)$ and hence α_i is an element of $\alpha_i A(y)$. Thus contradicting the fact$\alpha_i A(y) \cap \alpha_j A(y) = \emptyset$.

Therefore this corresponding is 1 – 1, for any object y' in Y, we have $\alpha(y) = y'$ for some permutation α in A. From the coset decomposition of$A$, it follows that

$\alpha = \alpha_i A(y)$ with y in $A(y)$

Hence $y' = \alpha_i(y)$ and thus every element of Y corresponds to some coset. Therefore m is the number of elements in Y and (4.1) is proved i.e.

$|A| = |A(y)||Y|$

Now we prepared for the first lemma which proves a formula for the number $N(A)$ of orbits of A in terms of the average number of fixed points of the permutation in A.

Lemma 4.1 (Burnnside's Lemma)

The number $N(A)$ of orbits of A. is given by

$N(A) = |A|^{-1} \sum_{\alpha \in A} J_i(\alpha)$ ….. (4.3)

Proof
Let $X_1, X_2, X_3, \dots, X_m$ be the orbits of A. for each $i = 1\ to\ m,$ let x_i be an element of the i^{th} orbit X_i. Then from (4.1) we have

$N(A)|A| = \sum_{i=1}^{m} |A(x_i)|\,|X_i|$ ….. (4.4)

We have seen that if $x\ and\ x_i$ are in the same orbits, the$A(x) = |A(x_i)|$. Hence the right side of (4.4) can be altered to obtain

$N(A)|A| = \sum_{x \in X} |A(x)|$

or in other notation , we have

$N(A)|A| = \sum_{x \in X} \sum_{\alpha \in A(x)} 1$ …. (4.5)

Now on interchanging the order of summation on the right of (4.5) and modifying the summation indices accordingly, we have

$N(A)|A| = \sum_{x\in X}\sum_{x\in \alpha x)} 1$

But $\sum_{x=\alpha x)} 1$ is just $j_i(\alpha)$. Thus the proof is completed on division by $|A|$

$N(A) = |A|^{-1}\sum_{\alpha\in A} j_i(\alpha)$.

Example

Consider a graph G below using the product notation, the group G may be expressed as $\Gamma(G) = S_1^3 S_7^2(\alpha_4)$. Now $\Gamma(G)$ has order 4 and each permutation fixes the three points 3,5 and 7

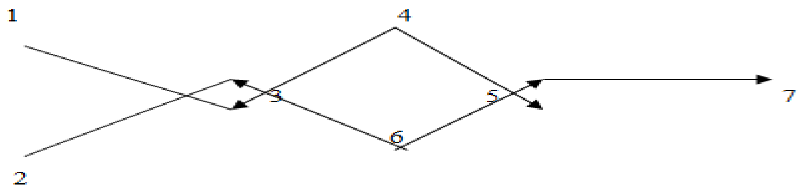

Figure 4.1 : A graph with three fixed points

Let the permutation be denoted by

$\alpha_1 = (1)(2)(3)(4)(5)(6)(7)$ $\alpha_2 = (12)(3)(4)(5)(6)(7)$

$\alpha_3 = (46)(1)(2)(3)(5)(7)$ $\alpha_4 = (12)(46)(3)(5)(7)$

Then $J_i(\alpha_1) = 7$, thus is all the seven points are fixed, $J_i(\alpha_2) = 5, J_i(\alpha_3) = 5, J_i(\alpha_4) = 3$

Thus $N(\Gamma(G)) = \frac{1}{4}(7+5+5+3) = 5$

And is clear that the orbits are $\{3\},\{5\},\{7\},\{1,2\}$ *and* $\{4,6\}$

The number of orbits is precisely the number of ways in which G can be rooted. To obtain all rooting of G one simply chooses one point from each of the orbits as a root.

Definition 4.1

Let G be a group of permutation of a set S, for each $s \text{ in } S \text{ i.e } s \in S. Orb_G(s) = \{S\alpha \mid \alpha \in G\}$

The set $Orb_G(s)$ is called a subset of a the orbits the orbits of s under G

Example on orbits

On the example on figure 4.1

$Orb_G(1) = \{1,2\}$ $Orb_G(2) = \{1,2\}$ $Orb_G(3) = \{3\}$

$Orb_G(4) = \{4,6\}$ $Orb_G(5) = \{5\}$ $Orb_G(6) = \{4,6\}$

$Orb_G(7) = \{7\}$

Therefore $Orb_G(s) = \{1,2\}.\{3\},\{4,6\},\{5\},\{7\}$

(2) Let {(1), (123)(67) ,(234)(56), (132)(456)(78), (78)}

$Orb_G(1) = \{1,2,3\}$ $Orb_G(2) = \{1,2,3\}$ $Orb_G(3) = \{1,2,3,4\}$

$Orb_G(4) = \{2,4,5\}$ $Orb_G(5) = \{5,6\}$ $Orb_G(6) = \{4,5,6,7\}$

$Orb_G(7) = \{6,7,8\}$ $Orb_G(8) = \{7,8\}$

These implies that $J_i(\alpha_1) = 8$, $J_i(\alpha_2) = 3,$ $J_i(\alpha_3) = 3,$ $J_i(\alpha_4) = 0$, $J_i(\alpha_5) = 6$

$$N(\Gamma(G)) = \frac{1}{5}(8 + 3 + 3 + 6 + 0) = \frac{20}{5} = 4$$

Therefore the orbits are {{1,2,3,4},{2,4,5}.{4,5,6,7}.{6,7,8}}

Example on stabilizer

Given a group ={(1), (123)(67) ,(234)(56), (132)(456)(78), (78)}

Therefore stabilizers of G are

$Stab_G(1) = (1), (234)(56), (78)$ $Stab_G(2) = (2), (78)$ $Stab_G(3) = (1), (78)$

$Stab_G(4) = (1), (123)(67), (78)$ $Stab_G(5) = (1), (123)(67), (78)$

$Stab_G(6) = (1), (78)$ $Stab_G(7) = (1), (234)(56)$

$Stab_G(8) = (1), (123)(67), (234)(56)$

That is all the points are fixed

$Stab_G(s) = \{(1), (78)\}$ $\{(1), (123)(67), (78)\}$ $\{(1)(234)(56), (78)\}$ $\{(1), (123)(67),$ $(234)(56)\}$ {(1),(234)(56)}= 5

Theorem 4.2

Let G be a finite group of permutation of a set S, then for any j of s, then

$|G| = |Orb_G(S)||Stab_G(S)|$ and is called the orbit-stabilizer theorem.

Example

For $S_2 = \{(1)(2),\ (12)\}$

$Stab_G(1) = (1)(2),$ $Stab_G(2) = (1)(2)$

$Orb_G(1) = (1)(2), (12)$ $Orbi_G(2) = (1)(2),$ (12)

$Stab_G(1) = (1)(2)$ and $Orb_G(1) = (1)(2),$ (12)

Hence $|G| = |Orb_G(S)||Stab_G(S)| = 1 \times 2 = 2$

For $S_3 = \{(1)(2)(3),\ (1)(23),\ (2)(13),\ (12)(3),\ (123),\ (132)$

$Stab_G(1) = (1)(2)(3), (1)(23)$ $Stab_G(2) = (1)(2)(3), (2)(13)$

$Stab_G(3) = (1)(2)(3), (12)(3)$

Also $Orb_G(1) = (1,2,3)$ $Orb_G(2) = (1,2,3)$ $Orb_G(3) = (1,2,3)$

Hence $|G| = |Orb_G(S)||Stab_G(S)| = 3 \times 2 = 6$

4.5. Weighted form of burnside lemma

We provided a slight generalization of (4.3) called the weighted form of burnside lemma

$N(A/Y) = A^{-1}\sum\ J_i(\alpha/Y)$ … (4.6)

Let R be any commutative ring containing the rational and let W be a function the weighted function, from the object set X of A into the ring R. In practice the weight function is constant on the orbits of A. Hence in this case we can define the weight of any orbit X to be the weight of any element in the orbit.

For each orbit X_i, we denote the weight of X_i by $w(X_i)$ and by definition $w(X_i) = w(x)$ for any $x \, in \, X$

Theorem 4.3.
The sum of the weights of the orbits of A is given by

$$\sum_{i=1}^{m} w(X_i) = |A|^{-1} \sum_{x=\alpha x} w(x) \quad \ldots (4.7)$$

The proof is similar to the proof of (4.3)

Consider the graph G in figure (4.1)

To illustrate this lemma and to display a sum of cycle indices in a way which will be used effectively for each point K of G. We define the weight $w(k)$ to be the cycle index of the stabilizer of K in $\Gamma(G)$. Thus

$$w(1) = \frac{1}{2}(s_1^7 + s_1^5 s_2) \quad \ldots (4.8)$$

i.e $Stab_G(1)$ has two stable points (1), (46)(1)(2)(3)(5)(7) which implies that the identity of length 1 is 7, and the cycle of length 1 and 2 are $s_1^5 s_2$).

$$w(3) = \frac{1}{4}(s_1^7 + 2s_1^5 s_2 + S_2^2 S_1^3) \quad \ldots .(4.9)$$

$Stab_G(3) = (1)(12)(3)(4)(5)(6)(7),\ (46)(1)(2)(3)(5)(7),\ (12)(46)(3)(5)(7),$

Note that $w(1) = w(2) = w(3) = w(6) \; i.e$

$w(2) = \frac{1}{2}(s_1^7 + s_1^5 s_2)$ $w(4) = \frac{1}{2}(s_1^7 + s_1^5 s_2)$ $w(6) = \frac{1}{2}(s_1^7 + s_1^5 s_2)$

Also $w(3) = w(5) = w(7)$ i.e

$w(3) = \frac{1}{4}(s_1^7 + 2s_1^5 s_2 + s_2^2 s_3)$ $w(5) = \frac{1}{4}(s_1^7 + 2s_1^5 s_2 + s_2^2 s_3)$
$w(7) = \frac{1}{4}(s_1^7 + 2s_1^5 s_2 + s_2^2 s_3)$

Thus in particular on the orbits.

We sketch the verification of (4.7) for this example by observing that the sum of the orbit weight is $w(1) + w(3) + w(4) + w(5) + w(7) = 2w(1) + 3w(3)$

Since $w(1) = w(4) \; and \; w(3) = w(5) = w(7)$

Since the right side of (4.7) is the sum

$$\frac{1}{4}\sum_{i=1}^{4}\sum_{x=\alpha_i x} w(x) = \frac{1}{4}\{\sum_{i=1}^{4}\sum_{x=\alpha_i x} w(x) + \ldots + \sum_{x=\alpha_i x} w(x)\}$$

$$= \frac{1}{4}\{\sum_{x=\alpha_i x} w(x) + \sum_{k=3}^{7} w(k) + \ldots\}$$

Where the last two terms in the above equation are found at once by inspection α_3 and α_4 in the graph of figure (4.1) and the verification is finished.

Similarly the cycle index sum for all the different rooted graphs obtained from any graph G can be obtained in terms of the weight of the fixed points $\Gamma(G)$.

V. CONCLUSION

In this research, it was discovered that the cycle index of symmetric group can be understand better by the use of formulas. The use of formula and examples was also used to establish that there is strong relationship between orbits of a symmetric group and orbits stabilizer of a symmetric group.

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