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Fixed Point Theorem in Fuzzy Metric Space by Using New Implicit Relation

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Abstract
In this paper we give a fixed point theorem on fuzzy metric space with a new implicit relation. Our results extend and generalize the result of Mishra and Chaudhary [10]

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I. Introduction

Zadeh [11] introduced the concept of fuzzy sets in 1965 and in the next decade Kramosil and Michalek [12] introduced the concept of fuzzy metric spaces in 1975, which opened an avenue for further development of analysis in such spaces.Vasuki [13] investigated same fixed point theorem s in fuzzy metric spaces for R-weakly commuting mappings and pant [14] introduced the notion of reciprocal continuity of mappings in metric spces.Balasubramaniam et aland S. Muralishankar,R.P. Pant [15] proved the poen problem of Rhodes [16]on existence of a contractive definition.

II. Preliminaries

Definition 2.1 [1] A binary operation * : [0,1] ×[0,1] →[0,1] is continuous t-norm if satisfies the following conditions:

(1) * is commutative and associative,
(2) * is continuous,
(3) a*1 = a for all a ϵ [0,1],
(4) a *b ≤ c*d whenever a ≤ c and b ≤ d for all a, b, c ,d ϵ[0,1].

Examples of t- norm are $a * b = min\{a, b\}$ *and* $a * b = ab$
Definition 2.3 [3] A 3-tuple $(X, M,*)$ is said to be a fuzzy metric space if X is an arbitrary set,* is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions:
The functions M(x ,y, t) denote the degree of nearness between x and y with respect to t , respectively .

1) $M(x, y, 0) = 0$ for all $x, y \in X$
2) $M(x, y, t) = 1$ *for all* $x, y \in X$ *and* $t > 0$ *if and only if* $x = y$
3) $M(x, y, t) = M(y, x, t)$ *for all* $x, y \in X$ *and* $t > 0$
4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$ *for all* $x, y, z \in X$ *and* $s, t > 0$,
5) *for all* $x, y \in X$, $M(x, y, .) : [0, \infty) \to [0,1]$ *is left continuous*,

Remark 2.1 In a FM $(X, M,*)$, $M(x, y, .)$ is non- decreasing for all x , $y \in X$.
Definition 2.4 Let $(X, M,*)$, be a FM - space. Then
(i) A sequence $\{x_n\}$ in X is said to be Cauchy Sequence if for all $t > 0$ and $p > 0$,
$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$
(ii) A sequence$\{x_n\}$ in X is said to be convergent to a point $x \in X$ if for all $t > 0$
$\lim_{t\to\infty} M(x_n, x, t) = 1$
Since * is continuous, the limit is uniquely determined from (5) and (11) repectively.

Definition 2.5 [11] A FM-Space $(X, M,*,)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 2.6 [4] Let A and S be maps from a fuzzy metric $(X, M, *,)$ into Itself .The maps A and S are said to be weakly commuting if

$$M(ASz, SAz, t) \geq M(Az, Sz, t) \quad \text{for all } z \in X \text{ and } t > 0$$

Definition 2.7 [6] Let A and S be maps from an FM-space$(X, M, *)$ into itself .The maps A and S are said to be compatible if for all $t > 0$ $\lim_{t\to\infty} M(ASx_n, SAx_n, t) = 1$ whenever$\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 2.8 [8] Two mappings A and S of a fuzzy metric space $(X, M, *)$ will be called reciprocally continuous if $ASu_n \to Az$, *and* $SAu_n \to Sz,$ whenever $\{u_n\}$ is a sequence such that for some $Au_n, Su_n \to z$ *for some* , $z \in X$

Definition 2.9 Let $(X, M, *)$ be a fuzzy metric space. A and S be self maps on X. A point x in X is called a coincidence point of A and S iff Ax =Sx. In this case w = Ax =S x is called a point of coincidence of A and S .

Definition 2.10 A pair of mappings (A,S) of a fuzzy metric space $(X, M, *)$ is said to be weakly compatible if they commute at the coincident points i.e., if Au = Su for some u in X then ASu = SAu.

Definition 2.11 [7] Two self maps A and S of a fuzzy metric space $(X, M, *)$ are said to be occasionally weakly compatible (owc) iff there is a point x in X which is coincidence point of A and S at which A and S commute.

Definition 2.12 (Implicit Relation) Let $\emptyset_5$ be the set of all real and continuous function from$(R^+)^5 \to R$ and such that

2.12 (i) $\emptyset$ is non increasing in $2^{nd}, 3^{rd}$ *and* 4^{th} argument and

2.12 (ii) for $u, v \geq 0$ $\emptyset(u, v, v, v, v) \geq 0 \Rightarrow u \geq v$

Example $\emptyset(t_1, t_2, t_3, t_4, t_5) = t_1 - max\{t_1, t_2, t_3, t_4\}$

Lemma 2.1 Let $\{u_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ .If there exist a constant $k \in (0,1)$ such that $M(u_n, u_{n+1}, kt) \geq M(u_{n-1}, u_n, t)$ for all $t > 0$ and n = 1 , 2 , 3….. Then $\{u_n\}$ is a Cauchy sequence in X .

Lemma 2.2 Let $(X, M, *)$ be a FM space and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0,1)$, $M(x, y, kt) \geq M(x, y, t)$ then x = y

Lemma 2.3 [9] Let X be a set , f and g be owc self maps of X. If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.

3.Main Result

Theorem 3.1 Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc . If for $\emptyset \in \emptyset_5$ there exist $q \in (0,1)$ such that

$$\emptyset\begin{pmatrix} M(Ax, By, qt), M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ \left(\dfrac{1 + M(Ax, Sx, t)}{1 + M(By, Ty, t)}\right) \cdot \left[\dfrac{M(Ax, Ty, t) + M(By, Sx, t)}{2}\right] \end{pmatrix} \geq 0 \qquad \text{………..(1)}$$

for all $x, y \in X$and $t > 0$,then there exist a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Morever $z = w$, so that there is a unique common fixed point of A,B, S and T

Proof Let the pairs $\{A,S\}$ and $\{B,T\}$ be owc, so there are points $x,\ y \in X$ such that $Ax = Sx \text{ and } By = Ty$. We claim that $Ax = By$. If not by inequality (1)

$$\emptyset\begin{pmatrix} M(Ax,By,qt), M(Ax,By,t), M(Sx,Ax,t), M(By,Ty,t), \\ \left(\frac{1+M(Ax,Sx,t)}{1+M(By,Ty,t)}\right).\left[\frac{M(Ax,Ty,t)+M(By,Sx,t)}{2}\right] \end{pmatrix} \geq 0$$

$$\emptyset\begin{pmatrix} M(Ax,By,qt), M(Ax,By,t), M(Ax,Ax,t), M(By,By,t), \\ \left(\frac{1+M(Ax,Ax,t)}{1+M(By,By,t)}\right).\left[\frac{M(Ax,By,t)+M(By,Ax,t)}{2}\right] \end{pmatrix} \geq 0$$

$$\emptyset\big(M(Ax,By,qt), M(Ax,By,t), 1,1, M(Ax,By,t)\big) \geq 0$$

$$\emptyset\big(M(Ax,By,qt), M(Ax,By,t), M(Ax,By,t), M(Ax,By,t), M(Ax,By,t)\big) \geq 0$$

$\because \emptyset \text{ is non} - \text{increasing in } 3^{rd} \text{ and } 4^{th}$ argument therefore 2.12 (i) and 2.12 (ii)

$M(Ax,By,qt) \geq M(Ax,By,t)$

Therefore $Ax = By$ $i.e.\ Ax = Sx = By = Ty$

Suppose that there is a unique point z such that $Az = Sz$ then by (1) we have

$$\emptyset\begin{pmatrix} M(Az,By,qt), M(Az,By,t), M(Sz,Az,t), M(By,Ty,t), \\ \left(\frac{1+M(Az,Sz,t)}{1+M(By,Ty,t)}\right).\left[\frac{M(Az,Ty,t)+M(By,Sz,t)}{2}\right] \end{pmatrix} \geq 0$$

$$\emptyset\big(M(Az,By,qt), M(Az,By,t), 1,1, M(Az,By,t)\big) \geq 0$$

$$\emptyset\big(M(Az,By,qt), M(Az,By,t), M(Az,By,t), M(Az,By,t), M(Az,By,t)\big) \geq 0$$

$\emptyset \text{ is non} - \text{increasing in } 3^{rd} \text{ and } 4^{th}$ argument therefore by 2.12 (i) and 2.12 (ii)

$M(Az,By,qt) \geq M(Az,By,t)$

$Az = By$ =Sz $= Ty$, So$Ax = Az \text{ and } w = Ax = Sx$ the unique point of coincidence of A and S. By lemma (2.3) w is the only common fixed point of A and S. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$

Assume that $w \neq z$ we have

$$\emptyset\begin{pmatrix} M(Aw,Bz,qt), M(Aw,Bz,t), M(Sw,Aw,t), M(Bz,Tz,t), \\ \left(\frac{1+M(Aw,Sw,t)}{1+M(Bz,Tz,t)}\right).\left[\frac{M(Aw,Tz,t)+M(Bz,Sw,t)}{2}\right] \end{pmatrix} \geq 0$$

$$\emptyset\begin{pmatrix} M(Aw,Bz,qt), M(w,z,t), M(w,z,t), M(z,z,t), \\ \left(\frac{1+M(w,w,t)}{1+M(z,z,t)}\right).\left[\frac{M(w,z,t)+M(z,w,t)}{2}\right] \end{pmatrix} \geq 0$$

$$\emptyset\big(M(Aw,z,qt), M(w,z,t), M(w,w,t), M(z,z,t), M(w,z,t)\big) \geq 0$$

$$\emptyset\big(M(Aw,z,qt), M(w,z,t), 1,1, M(w,z,t)\big) \geq 0$$

$$\emptyset\big(M(Aw,z,qt), M(w,z,t), M(w,z,t), M(w,z,t), M(w,z,t)\big) \geq 0$$

$\because \emptyset$ is non increasing in 3^{rd} and 4^{th} argument

$\therefore M(Aw,Bz,qt), \geq M(w,z,t)$

We have z = w by Lemma (2.2) and z is a common fixed point of A , B , S and T .The uniqueness of the fixed point holds from (1).

Definition 3.11 (Implicit Relation) Let $\emptyset_6$ be the set of all real and continuous function from $(R^+)^6 \to R$ and such that

3.11 (i) $\emptyset$ is non increasing in $2^{nd}, 3^{rd}\ 4^{th}$ *and* 5^{th} argument and

3.11 (ii) for $u, v \geq 0\ \ \emptyset(u, v, v, v, v, v) \geq 0 \Rightarrow u \geq v$ *and* $\psi(u, v, v, v, v, v) \geq 0 \Rightarrow u \leq v$

Theorem 3.2 Let $(X, M, *)$ be a complete fuzzy metric space and let A , B , S and T be self mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc . If there exist $q \in (0,1)$ such that

$$\emptyset\left(\begin{array}{c} M(Ax, By, qt), M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ \left(\frac{1 + M(Ax, Sx, t)}{1 + M(By, Ty, t)}\right) \cdot \left[\frac{M(Ax, Ty, t) + M(By, Sx, t)}{2}\right], M(By, Sx, t) \end{array}\right) \geq 0$$

……….(2)

Proof Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc, so there are points $x,\ y \in X$ such that $Ax = Sx$ *and* $By = Ty$ We claim that $Ax = By$. If not by inequality (2)

$$\emptyset\left(\begin{array}{c} M(Ax, By, qt), M(Ax, By, t), M(Ax, Ax, t), M(By, By, t), \\ \left(\frac{1 + M(Ax, Sx, t)}{1 + M(By, Ty, t)}\right) \cdot \left[\frac{M(Ax, By, t) + M(By, Ax, t)}{2}\right], M(By, Ax, t) \end{array}\right) \geq 0$$

$$\emptyset\left(\begin{array}{c} M(Ax, By, qt), M(Ax, By, t), 1, 1, \\ M(Ax, By, t), M(By, Ax, t) \end{array}\right) \geq 0$$

$$\emptyset\left(\begin{array}{c} M(Ax, By, qt), M(Ax, By, t), M(Ax, By, t), M(Ax, By, t), \\ M(Ax, By, t), M(By, Ax, t) \end{array}\right) \geq 0$$

$\because \emptyset$ *is non − increasing in* 3^{rd} *and* 4^{th} argument therefore by 3.11(i) and 3.11(ii)

$$M(Ax, By, qt) \geq M(Ax, By, t)$$

Therefore $Ax = By$ *i.e.* $Ax = Sx = By = Ty$. Suppose that there is another point z such that A z =S z then by (2) we have A z = S z = Ty, So Ax = A z and w =Ax = Tx is the unique point of coincidence of A and T. By lemma(2.2) w is a unique point $z \in X$ such that $z = Bz = Tz$. Thus z is a common fixed point of A ,B , S and T . The uniqueness of fixed point holds by (2).

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